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1. The Real Numbers

We will take the view that we know what the real numbers are and in this section we simply *review* some important properties.

The following notations for the *natural numbers*, *integers* and *rational numbers*, respectively.

$$\begin{aligned}\mathbb{N} =& \{0, 1, 2, \dots\}\\ \mathbb{Z} =& \{0, \pm 1, \pm 2, \dots\}\\ \mathbb{Q} =& \{\frac{m}{n} : m \in \mathbb{Z}, \quad n \in \mathbb{N}^+\},\end{aligned}$$

will be used, where $\mathbb{N}^+$ is the set of nonzero elements of $\mathbb{N}$. Let $\mathbb{R}$ denote the real numbers.

Example 1.1. The square root of 2 is not rational; i.e., there is no real number $s > 0$ such that $s^2 = 2$.

1.1. Field Axioms.

Definition 1.2. A *field* $\mathbb{F}$ is a triple, $(\mathbb{F}, +, \cdot)$, where $\mathbb{F}$ is a set and

$$+, \cdot : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$

are functions, called addition and multiplication respectively and written $x + y = +(x, y)$ and $xy = \cdot(x, y)$, satisfying the following (long list) of axioms

(i) $x + y = y + x$ for every $x, y \in \mathbb{F}$;
(ii) $xy = yx$, for every x, y;
(iii) $(x + y) + z = x + (y + z)$ for every x, y, z;
(iv) $(xy)z = x(yz)$ for every x, y, z;
(v) there is an element $0 \in \mathbb{F}$ such that $0 + w = w$ for every $x \in \mathbb{F}$;
(vi) there is an element $1 \in \mathbb{F}$, distinct from 0, such that $1w = w$ for every $w \in \mathbb{F}$;
(vii) for each $x \in \mathbb{F}$ there is an element $u \in \mathbb{F}$ such that $x + u = 0$;
(viii) for each $x \neq 0$, there is a y such that $xy = 1$; and
(ix) $(x + y)z = xz + yz$ for every x, y, z.

Proposition 1.3 (Cancellation)**.** *Given $x, y, z \in \mathbb{F}$, if $x + y = x + z$, then $y = z$.*

Proof. There exists $u \in \mathbb{F}$ such that $x + u = 0$. Thus,

$$\begin{aligned} y =& 0 + y \\ =&(u + x) + y \\ =&u + (x + y) \\ =&u + (x + z) \\ -&(u + x) + z \\ =&0 + z = z. \end{aligned}$$

□

Remark 1.4. It follows that 0 and additive inverses are unique. Hence it makes sense to write $u = -x$ in case $x + u = 0$ so that $x + (-x) = 0$.

Proposition 1.5. *Given* $x \in \mathbb{F}$, $0x = 0$ *and* $-x = (-1)x$.

Proof. Since $0 + 0x = 0x = (0 + 0)x = 0x + 0x$, cancellation gives $0 = 0x$.
Using $0x = 0$ gives $x + (-1)x = 1x + (-1)x = (1 + (-1))x = 0x = 0$. □

Remark 1.6. From here on we will use freely, without proof or further comment, the many routine properties of fields which follow from the axioms.

Example 1.7. The sets $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all fields with their usual operations of addition and multiplication.

Example 1.8. $\mathbb{Q}(\sqrt{2})$ is a field.

Example 1.9. Let $\mathbb{Z}_2 = (\{0, 1\}, +, \cdot)$ where

$$\begin{aligned} x + y &= x + y \text{ modulo } 2 \\ xy &= xy \text{ modulo } 2 \end{aligned}$$

Here the $+$ on the left hand side is addition in $\mathbb{Z}_2$, whereas $+$ on the right hand side is addition in $\mathbb{N}$.

The residue modulo 2 is the remainder after dividing by 2.

It is easy, but tedious, to verify that $\mathbb{Z}_2$ is a field with neutral elements $0, 1$.

Example 1.10. $\mathbb{Z}$ is not a field. The smallest field containing $\mathbb{Z}$ is $\mathbb{Q}$.

1.2. **Ordered Fields.**

Definition 1.11. An *ordered set* $(S, <)$ consists of a (nonempty) set S and a relation $<$ on S which satisfies the *trichotomy*; i.e., for each $x, y \in S$, exactly one of the following hold,

$$x < y, \quad y < x, \quad x = y;$$

and is *transitive*; i.e., if $x < y$ and $y < z$, then $x < z$.

Example 1.12. The usual order on $\mathbb{R}$ (and thus on any subset of $\mathbb{R}$) is an example of an ordered set. In particular $\mathbb{Q}$ is an order set and so is $\mathbb{Q}(\sqrt{2})$.

Definition 1.13. An *ordered field* $\mathbb{F} = (\mathbb{F}, +, \cdot, <)$ consists of a field $(\mathbb{F}, +, \cdot)$ which is also an ordered set $(\mathbb{F}, <)$ such that,

(i) if $x, y, z \in \mathbb{F}$ and $x < y$, then $x + z < y + z$;
(ii) if $x, y \in \mathbb{F}$ and $x, y > 0$, then $xy > 0$.

An element $x \in \mathbb{F}$ is *positive* if $x > 0$.

Example 1.14. Both $\mathbb{R}$ and $\mathbb{Q}$ with the usual ordering are ordered fields.

Proposition 1.15. Suppose $\mathbb{F}$ is an ordered field and $x \in \mathbb{F}$. If $x < 0$, then $-x > 0$. If $x \neq 0$, then $x^2 > 0$. In particular, $1 > 0$ in any ordered field.

Proof. If $x < 0$, then $0 = x - x < 0 - x$.

If $x > 0$, then $x^2 = xx > 0$.

If $x < 0$, then $-x > 0$ and thus $x^2 = (-x)^2 > 0$. ☐

Remark 1.16. We will not state (much less) prove the usual facts about the order structure in an ordered field, but rather use them without comment.

Example 1.17. Prove that there is no order on $\mathbb{Z}_2$ which makes it an ordered field.

Arguing by contradiction, suppose $<$ is an order on $\mathbb{Z}_2$ which makes $\mathbb{Z}_2$ an ordered field. Since $1 = 1^2$, it follows that $1 > 0$ and hence $-1 < 0$. On the other hand, $-1 = 1 > 0$, a contradiction (of trichotomy).

1.3. The least upper bound property.

Definition 1.18. Let S be a subset of an ordered field $\mathbb{F}$.

(i) The set S is *bounded above* if there is a $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$.
(ii) Any $b \in \mathbb{F}$ such that $b \geq s$ for all $s \in S$ is an *upper bound* for S.

Example 1.19. Identify the set of upper bounds for the following subsets of the ordered field $\mathbb{R}$.

(a) $[0, 1)$;
(b) $[0, 1]$;
(c) $\mathbb{Q}$;
(d) $\emptyset$.

Lemma 1.20. *Let S be a subset of an ordered field $\mathbb{F}$ and suppose both b and b' are upper bounds for S. If b and b' both have the property that if $c \in \mathbb{F}$ is an upper bound for S, then $c \geq b$ and $c \geq b'$, then $b = b'$.*

Definition 1.21. The *least upper bound* for a subset S of an ordered field $\mathbb{F}$, if it exists, is a $b \in \mathbb{F}$ such that

(i) b is an upper bound for S; and
(ii) if $c \in \mathbb{F}$ is an upper bound for S, then $c \geq b$.

Remark 1.22. Lemma 1.20 justifies the use of *the* (as opposed to *an*) in describing the least upper bound.

The condition (ii) can be replaced with either of the following conditions

(ii)′ if $c < b$, then there exists an $s \in S$ such that $c < s$; or
(ii)″ for each $\epsilon > 0$ there is an $s \in S$ such that $b - \epsilon < s$.

The notions of *bounded below, lower bound* and *greatest lower bound* are defined analogously.

Least upper bound is often abbreviated lub. The term *supremum*, often abbreviated *sup*, is synonymous with lub. Likewise glb and *inf* for greatest lower bound and infimum.

Example 1.23. Here is a list of examples.

(i) The least upper bound of $S = [0, 1) \subset \mathbb{R}$ is 1.
(ii) The least upper bound of $V = [0, 1] \subset \mathbb{R}$ is also 1.
(iii) The set $\mathbb{Q} \subset \mathbb{R}$ has no upper bound and thus no least upper bound;
(iv) Every real number is an upper bound for the set $\emptyset \subset \mathbb{R}$. Thus $\emptyset$ has no least upper bound.
(v) With some effort, it can be shown that the least upper bound of the subset $\{x \in \mathbb{Q} : 0 < x, \quad x^2 < 2\} \subset \mathbb{R}$ is $\sqrt{2}$.

Example 1.24. Consider the subset $S = \{q \in \mathbb{Q} : 0 < q, \quad q^2 < 2\}$ of the ordered field $\mathbb{Q}$. With some effort, it is possible to show that S has no least upper bound: By way of contradiction, one shows, as in the previous example, if $b \in \mathbb{Q}$ were a least upper bound for S, then $0 < b$ and $b^2 = 2$ which is a contradiction. (See Example 1.1.)

Theorem 1.25. Every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound.

Remark 1.26. In fact, $\mathbb{R}$ is essentially the unique ordered field satisfying the conclusion of Theorem 1.25. This property, which thus distinguishes $\mathbb{R}$ from all other ordered fields, is a a completeness property.

For comparison, $\mathbb{Q}$ is an ordered field. However,the set

$$S = \{q \in \mathbb{Q} : q > 0, \quad q^2 < 2\} \subset \mathbb{Q}$$

is non-empty and bounded above (by 2), but does not have a least upper bound (in $\mathbb{Q}$). See item (v) of Example 1.23.

Theorem 1.27 (Archemedian properties of $\mathbb{R}$)**.** *Suppose $x, y \in \mathbb{R}$.*

(i) *There is a natural number n so that $n > x$;*
(ii) *If $y > 0$, then there is an $n \in \mathbb{N}^+$ such that $\frac{1}{n} < y$; and*
(iii) *If $x < y$, then there is a $q \in \mathbb{Q}$ such that $x < q < y$.*

Remark 1.28. The last part of the theorem is sometimes expressed as saying $\mathbb{Q}$ is *dense* in $\mathbb{R}$.

Proof. To prove (i), by arguing by contradiction, suppose no such natural number exists. In that case x is an upper bound for $\mathbb{N}$. It follows that $\mathbb{N}$ has a lub α. If $n \in \mathbb{N}$, then $n + 1 \leq \alpha$ and thus $n \leq \alpha - 1$ for all $n \in \mathbb{N}$. Thus, $\alpha - 1$ is an upper bound for $\mathbb{N}$, contradicting the least property of α. Hence $\mathbb{N}$ is not bounded above and there is an $n > x$, which proves item (i).

Item (ii) follows by applying (i) to $x = \frac{1}{y}$.

To prove (iii), choose $n \in \mathbb{N}^+$ so that $1 < n(y-x)$. Choose $m \in \mathbb{Z}$ so that

$$m - 1 \leq nx < m.$$

Rearranging the inequalities gives,

$$nx < m < nx + 1 < ny.$$

Hence $x < \frac{m}{n} < y$. □

Example 1.29. Suppose $0 < a < 1$. Show the set $A = \{a^n : n \in \mathbb{N}\}$ is bounded below and its infimum is 0. Since $a \geq 0$ each $a^n \geq 0$. Thus A is bounded below by 0. The set A is not empty. It follows that A has an infimum. Let $\alpha = \inf(A)$ and note $\alpha \geq 0$. Since $\alpha \leq a^n$ for $n = 0, 1, 2, \ldots$, $\alpha \leq a^{n+1}$ for $n \in \mathbb{N}$ and therefore $\frac{\alpha}{a} \leq a^n$ for $n \in \mathbb{N}$. Thus, $\frac{\alpha}{a}$ is a lower bound for A. It follows that $\frac{\alpha}{a} \leq \alpha$. Since $a < 1$ and $\alpha \geq 0$, $\alpha = 0$.

1.4. **Accumulation points and the Balzano-Weierstrass Theorem.**

Definition 1.30. Let S be a given subset of $\mathbb{R}$. A point $a \in \mathbb{R}$ is an *accumlation point* (synonymously *limit point*) of S if for each $\epsilon > 0$ there is an $s \in S$ such that $0 < |a - s| < \epsilon$.

Example 1.31. The point 0 is an accumulation point of the set $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$.

The point 0 is also an accumulation point of the set $T = S \cup \{0\}$.

Lemma 1.32. *Suppose S, T are subsets of $\mathbb{R}$ and $a \in \mathbb{R}$.*

(i) If a is an accumulation point of S, then a is an accumulation point of $S \setminus \{a\}$.
(ii) If $S \subset T$ and a is an accumulation point of S, then a is an accumulation point of T.
(iii) The point a is an accumulation point of S if and only if for every $\epsilon > 0$ the set $(a - \epsilon, a + \epsilon) \cap S$ is infinite.

Example 1.33. (i) If $F \subset \mathbb{R}$ is finite, then F has no accumulation points.
(ii) The set of accumulation points of $S = \{\frac{1}{n} : n \in \mathbb{N}^+\}$ is exactly $\{0\}$; i.e., if $r \neq 0$, then r is not an accumulation point of S.
(iii) The set $\mathbb{Z}$ has no accumulation points.
(iv) The set of accumulation points of the set $(0, 1)$ is the set $[0, 1]$.
(v) The set of accumulation points of $\mathbb{Q}$ is $\mathbb{R}$, a fact which is equivalent to the statement that between any two real numbers there is a rational, and often expressed by saying the rationals are dense in the real numbers. See Theorem 1.27.

Theorem 1.34 (Balzano-Weierstrass)**.** *If S is an infinite and bounded subset of $\mathbb{R}$, then S has an accumulation point.*

Proof. Since S is bounded, there exists a $C > 0$ such that $S \subset [-C, C]$. Let

$$T = \{r \in \mathbb{R} : S \cap (-\infty, r] \text{ is finite}\}.$$

Note that $-C \in T$ since $S \cap (-\infty, -C] \subset \{C\}$. Thus T is nonempty. On the other hand, since T is infinite and a subset of $(-\infty, C]$, it follows that C is an upper bound for T. Hence T has a least upper bound α.

If $\beta < \alpha$, then there is a $\beta < \gamma < \alpha$ such that $\gamma \in T$ by the least property of α. In particular, $S \cap (-\infty, \gamma]$ is a finite set and thus so is $S \cap (-\infty, \beta]$. On the other hand, if $\delta > \alpha$, then $\delta \notin T$ as α is an upper bound for T. In particular, $S \cap (-\infty, \delta]$ is an infinite set. It follows that $S \cap (-beta, \delta]$ is an infinite set. Given $\epsilon > 0$, let $\beta = \alpha - \epsilon$ and $\delta = \alpha + \frac{\epsilon}{2}$ so that $S \cap (\alpha - \epsilon, \alpha + \epsilon)$ is infinite. Hence α is an accumulation point of S and the proof is complete. □

1.5. **The existence of n-th roots.** Here is an outline a proof that every positive real numbers have n-th roots for positive positive integers n.

Proposition 1.35. *If $y > 0$ and $n \in \mathbb{N}^+$, then there is a unique positive real number s such that $s^n = y$.*

The uniqueness is straightforward based upon the fact that if $0 < a < b$, then $a^n < b^n$. It should not come as a shock that existence depends upon the existence of least upper bounds, Theorem 1.25.

Let

$$S = \{x \in \mathbb{R} : 0 < x \text{ and } x^n < y\}.$$

First show that if $u > 0$ and $u^n \geq y$, then u is an upper bound for S. Next Argue that S is nonempty and bounded above. Hence S has a least upper bound, say s. If $0 < t$ and $t^n < y$, then there exists a $0 < t < u$ such that $u^n < y$. Hence, since s is an upper bound for S, it must be the case that either $s^n > y$ or $s^n = y$. On the other hand, show if $0 < t$ and $t^n > y$, then there is a $0 < u < t$ such that $u^n > y$. Hence, $u < t$ and u is an upper bound for S. Thus, $s^n \leq y$. By trichotomy, $s^n = y$.

1.6. **The Cauchy-Schwarz and triangle inequalities.** Let $\mathbb{R}^d$ denote the set of matrices of size $d \times 1$. Thus an element of $a \in \mathbb{R}^d$ has the form

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix},$$

for real numbers $a_1, \dots, a_d$.

Given $a, b \in \mathbb{R}^d$, define the *inner product* of a and b by

$$\langle a, b \rangle = \sum a_j b_j.$$

The inner product is also called the *dot product* and *scalar product*. The norm of $a \in \mathbb{R}^d$ is

$$\|a\| = \sqrt{\langle a, a \rangle}.$$

The interpretation of the dot product and norm should be familiar in the cases $d = 2$ and $d = 3$.

Proposition 1.36 (Cauchy-Schwarz inequality)**.** *Given* $a, b \in \mathbb{R}^d$,

$$|\langle a, b\rangle| \leq \|a\|\,\|b\|.$$

Proof. Consider, for $t \in \mathbb{R}$,

$$0 \leq \|a + tb\|^2 = \|a\|^2 + 2t\langle a, b\rangle + t^2\|b\|^2.$$

It follows that the discriminant of this polynomial is non-positive; i.e.,

$$|\langle a, b\rangle|^2 - \|a\|^2\,\|b\|^2 \leq 0.$$

□

Proposition 1.37. *If* $a, b \in \mathbb{R}^d$, *then*

$$\|a + b\| \leq \|a\| + \|b\|.$$

Proof. From

$$\|a + b\|^2 = \|a\|^2 + 2\langle a, b\rangle + \|b\|^2$$

and Proposition 1.36 it follows that

$$\|a + b\|^2 \leq \|a\|^2 + 2\|a\|\,\|b\| + \|b\|^2 = (\|a\| + \|b\|)^2.$$

□

1.7. **Problems.**

Problem 1.1. Let $\mathbb{Z}_3 = (\{0, 1, 2\}, +, \cdot)$ where

$$x + y = x + y \text{ modulo } 3$$
$$xy = xy \text{ modulo } 3$$

It is easy to verify that $\mathbb{Z}_3$ is a field.

Find the additive inverse for 1.

Show there is no order $<$ on $\{0, 1, 2\}$ such that $(\mathbb{Z}_3, <)$ is an ordered field. (Suggestion: Arguing by contradiction, show the additive inverse for 1 would have to be both positive and negative.)

Problem 1.2. See the wikipedia page on the field of *complex numbers* $\mathbb{C}$. Suppose $z \in \mathbb{C}$ is not zero.

(a) Show if and $z = x + iy$ is the rectangular representation of z, then

$$z^{-1} = \frac{\overline{z}}{|z|^2},$$

where $|z|^2 = x^2 + y^2$.

(b) Show, if $z = r(\cos(\theta) + i\sin(\theta))$ is the polar representation of z, then

$$z^{-1} = \frac{1}{r}(\cos(\theta) - i\sin(\theta)).$$

Interpret geometrically.

Problem 1.3. Show there is no order $<$ on $\mathbb{C}$ such that $(\mathbb{C}, <)$ is an ordered field. (Suggestion: Consider i^2.)

Problem 1.4. The *greatest lower bound* (*glb*) or *infimum* (*inf*) is defined by simply reversing the inequalities in the definition of least upper bound. Find, with proof, the least upper bound of the set

$$\{\frac{1}{n} : n \in \mathbb{N}^+\}.$$

Problem 1.5. Suppose $S \subset T \subset \mathbb{R}$. Show, if T is bounded above and S is non-empty, then both S and T have least upper bounds and moreover,

$$\sup(S) \le \sup(T).$$

Problem 1.6. Suppose $S \subset \mathbb{R}$ is non-empty and bounded above (and hence has a least upper bound). Given $a \in \mathbb{R}$, let

$$T = a + S := \{a + s : s \in S\}.$$

Prove that T is non-empty and bounded above and moreover,

$$\sup(T) = a + \sup(S).$$

Problem 1.7. Show, if S and T are both nonempty and bounded above, then so is

$$S + T = \{s + t : s \in S, \quad t \in T\}$$

and moreover,

$$\sup(S + T) = \sup(S) + \sup(T).$$

[Suggestion. Given $s \in S$, note that $\sup(s+T) \le \sup(S+T)$. On the other hand, by the previous problem, $\sup(s + T) = s + \sup(T)$. Proceed.]

Problem 1.8. Given a positive real number y and positive integers m and n, show

$$(y^{\frac{1}{n}})^m = (y^m)^{\frac{1}{n}}.$$

Thus, $y^{\frac{m}{n}}$ is unambiguously defined.

Problem 1.9. Verify the claims in Example 1.33.

Problem 1.10. Give an example of a set with exactly two accumulation points.

Problem 1.11. Let S' denote the set of accumulation points of a subset S of $\mathbb{R}$. Show, $(S')' \subset S'$.

Use the set S from Item (ii) of Example 1.33 to show that inclusion can be proper.

Show, if $\mathbb{Q} \subset S'$, then $S' = \mathbb{R}$.

As a challenge question: What about S'''?

Problem 1.12. Show, if a, b are non-negative real numbers, then

(i) $a < b$ if and only if $a^2 < b^2$; and
(ii) $ab \le \frac{a^2+b^2}{2}$.

Problem 1.13. Show, if $a, b \in \mathbb{R}^d$, then $|\|a\| - \|b\|| \le \|a - b\|$.

2. Sequences

Definition 2.1. A *sequence* is a function a whose domain is $\mathbb{N}$ or more generally a set of the form $\{n \in \mathbb{Z} : n \geq k\}$ for some $k \in \mathbb{Z}$. It is commonly denoted as $(a_n) = (a_n)_k^\infty$ where $a_n = a(n)$ is the value of a at n. In these notes, generally a is assumed to take real values so that each $a_n \in \mathbb{R}$.

Example 2.2. Here are a few examples of sequences.

(i) $(a_n = \frac{1}{n})_{n=1}^\infty$;
(ii) $(a_n = (-1)^n)$;
(iii) $(a_n = n)$.

2.1. Limits.

Definition 2.3. Suppose $A \in \mathbb{R}$ and (a_n) is a sequence of real numbers. The sequence (a_n) *converges to* A if for every $\epsilon > 0$ there is an N such that if $n \geq N$, then $|a_n - A| < \epsilon$. The notations $(a_n) \to A$ and

$$\lim a_n = \lim_n a_n = \lim_{n \to \infty} a_n = A$$

are short hand for the statement (a_n) converges to A.

The sequence (a_n) *converges* if there is an A to which it converges. Otherwise, the sequence *diverges*.

Proposition 2.4. If (a_n) converges to both A and B, then $A = B$.

Remark 2.5. In view of the proposition, A *the limit* of the sequence (a_n).

Example 2.6. Show that $(\frac{1}{n})$ converges to 0.

Given $\epsilon > 0$ choose, by the Archimedean property of $\mathbb{R}$ (Theorem 1.27), an $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \epsilon$. Now, if $n \geq N$, then,

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Example 2.7. Show that the sequence (a_n) defined by

$$a_n = \frac{n}{n+2}$$

converges to 1.

Given $\epsilon > 0$ choose, by the Archimedean property, $N \in \mathbb{N}^+$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Now, if $n \geq N$, then

$$|\frac{n}{n+2} - 1| = \frac{2}{n+2} \leq \frac{2}{N} < \epsilon.$$

Example 2.8. Show that the sequence (b_n) defined by

$$b_n = \frac{n^2+2}{2n^3 - n - 2}$$

converges to 0.

Given $\epsilon > 0$ choose, by the Archimedean property, N so that $N \geq \max\{2, \frac{2}{\epsilon}\}$. Now, if $n \geq N$, then

$$\begin{aligned}|b_n - 0| =& \frac{n^2+2}{2n^3-n-2}\\ \leq& \frac{2n^2}{2n^3-n-2}\\ \leq& \frac{2n^2}{n^3}\\ \leq& \frac{2}{N} < \epsilon.\end{aligned}$$

Example 2.9. Fix $0 \leq a < 1$ and let $a_n = a^n$ (for $n \geq 0$). To show that (a^n) converges to 0, recall that the greatest lower bound of the set $A = \{a^n : n \in \mathbb{N}\}$ is 0. In particular, given $\epsilon > 0$, there is an $b \in A$ such that $0 \leq b < \epsilon$. There is an N such that $b = a^N$. If $n \geq N$, then $0 \leq a^n \leq a^N = b < \epsilon$. Hence, if $n \geq N$, then $|0 - a^n| < \epsilon$ and thus (a^n) converges to 0.

Another proof that (a^n) converges to 0 is given in Example 2.20.

Here is a list of simple properties of limits.

Proposition 2.10. *Let $(a_n)_k^\infty$ be a sequence from $\mathbb{R}$.*

(a) The sequence $(a_n)_k^\infty$ converges if and only if for each $\ell > k$ the sequence $(a_n)_\ell^\infty$ converges;

(b) if there is an N and an ℓ such that for $n \geq N$, $b_n = a_{n+\ell}$, then (a_n) converges if and only if (b_n) converges and in that case they converge to the same value; and

(c) if (a_n) converges to A and $c \in \mathbb{R}$, then (ca_n) converges to cA.

2.2. **Cauchy Sequences.**

Definition 2.11. A sequence (a_n) is *Cauchy* if for every $\epsilon > 0$ there is an N so that if $m, n \geq N$, then $|a_n - a_m| < \epsilon$.

Proposition 2.12. *If (a_n) converges, then (a_n) is Cauchy.*

Proof. Let A denote the limit of the sequence (a_n). Let $\epsilon > 0$ be given. There is an N so that if $n \geq N$, then $|a_n - A| < \frac{\epsilon}{2}$. Hence, if both $m, n \geq N$, then

$$|a_n - a_m| \leq |a_n - A| + |A - a_m| < 2\frac{\epsilon}{2} = \epsilon.$$

□

Example 2.13. The sequence $((-1)^n)$ diverges.

Using the contra-positive of Proposition 2.12 it suffices to show there exists an $\epsilon_0 > 0$ such that for every N there exists $m, n \geq N$ such that $|a_n - a_m| \geq \epsilon_0$ (with $a_n = (-1)^n$).

Choose $\epsilon_0 = 1$. Given N, let $n = N$ and $m = N + 1$. Since m, n have different parities, $|(-1)^n - (-1)^m| = 2 \geq \epsilon_0 = 1$.

Definition 2.14. A sequence (a_n) is *bounded above* if the set $S = \{a_n : n \in \mathbb{N}\}$ is bounded above; i.e., there is an M so that $a_n \leq M$ for all n. It is *bounded* if it is bounded above and below; i.e., if there is an M so that $|a_n| \leq M$ for all n.

Proposition 2.15. *If (a_n) is Cauchy, then (a_n) is bounded. In particular, convergent sequences are bounded.*

The proof does not use anywhere near the full strength of the Cauchy condition.

Proof. With $\epsilon = 1$ there is an N so that if $n, m \geq N$, then $|a_n - a_m| < 1$. Hence, $|a_n - a_N| < 1$ for all $n \geq N$ and thus $|a_n| \leq |a_N| + 1$ for $n \geq N$. There is a C such that $|a_n| \leq C$ for all $n \leq N-1$. Hence, $|a_n| \leq |a_N| + 1 + C$ for all n. □

Example 2.16. The sequence $(a_n = n)$ diverges, since the set $\mathbb{N}$ is not bounded above by Theorem 1.27.

Theorem 2.17. *If (a_n) is Cauchy, then (a_n) converges.*

Proof. Let S denote the range of the sequence. Thus $S = \{a_n : n \in \mathbb{N}\}$. By Proposition 2.15, the set S is bounded. If S is finite, then there is an A such that $a_n = A$ for infinitely many n. In particular, the set $I = \{m : a_m = A\}$ is infinite. To prove (a_n) converges to A, let $\epsilon > 0$ be given. There is an N such that for $m, n \geq N$, $|a_n - a_m| < \epsilon$. Thus, if $n \geq N$, then, choosing $m \in I$ and $m \geq N$, it follows that $|A - a_n| = |a_m - a_n| < \epsilon$.

Now suppose S is infinite. Then, by Theorem 1.34, S has an accumulation point A. To prove that (a_n) converges to A, let $\epsilon > 0$ be given. There is an N such that if $m, n \geq N$, then $|a_n - a_m| < \frac{1}{2}\epsilon$. On the other hand, the set $(A-\epsilon, A+\epsilon) \cap S$ is infinite, so there is an $m \geq N$ such that $a_m \in (A-\epsilon, A+\epsilon)$; i.e., $|A - a_m| < \frac{1}{2}\epsilon$. Hence, if $n \geq N$, then

$$|A - a_n| \leq |A - a_m| + |a_m - a_n| < \epsilon.$$

□

2.3. **Monotone Sequences.**

Definition 2.18. The sequence (a_n) is *increasing* if $a_{n+1} \geq a_n$ for all n and it is *strictly increasing* if $a_{n+1} > a_n$ for all n. The notions of *decreasing* and *strictly decreasing* are defined analogously. A *monotone sequence* is one which is either increasing or decreasing.

Theorem 2.19. *If (a_n) is increasing and bounded above, then (a_n) converges.*

A sequence $(a_n)_k^\infty$ is *eventually monotone* if there is an M so that $(a_n)_M^\infty$ is monotone. Theorem 2.19 holds if (a_n) is assumed only eventually increasing. Indeed, in most cases, monotone can be replaced by eventually monotone in the statements to follow.

Proof. The set $S = \{a_n : n \in \mathbb{N}\}$ is non-empty and, by hypothesis, bounded above. Hence S has a supremum A. To show that (a_n) converges to A, let $\epsilon > 0$ be given. There is $s \in S$ such that $A - \epsilon < s$. There is an N so that $s = a_N$. Now, if $n \geq N$, then

$$0 \leq A - a_n \leq A - a_N = A - s < \epsilon.$$

□

Example 2.20. Fix $0 < r < 1$ and consider the sequence $(a_n = r^n)$. Since $a_{n+1} = ra_n$, the sequence is decreasing. It is also bounded below by 0. Hence (r^n) converges to some L. To see that $L = 0$, note that $(a_{n+1}) = (ra_n)$ and (a_{n+1}) converges to L; whereas, (ra_n) converges to rL. Thus, $L = rL$ and since $r \neq 1$, $L = 0$.

Example 2.21. Let $a_1 = \sqrt{2}$ and define, recursively, $a_{n+1} = \sqrt{2 + \sqrt{a_n}}$. The following induction argument shows that (a_n) is increasing.

First, note that $a_2 = \sqrt{2 + \sqrt{2}} \geq \sqrt{2} = a_1$. Now suppose that $a_n \geq a_{n-1}$. In this case, $\sqrt{a_n} \geq \sqrt{a_{n-1}}$ and hence,

$$a_{n+1} = \sqrt{2 + \sqrt{a_n}} \geq \sqrt{2 + \sqrt{a_{n-1}}} = a_n,$$

and the induction argument is complete.

An induction argument shows that (a_n) is bounded above by 2. Hence, by Theorem 2.19, the sequence (a_n) converges to some A.

Definition 2.22. The sequence (a_n) *diverges to* ∞ if for each $C > 0$ there is an N so that if $n \geq N$, then $a_n > C$.

Example 2.23. The sequence $(a_n = n)$ diverges to ∞.

To see that the sequence $(b_n = \sqrt{n})$ diverges to ∞, let $C > 0$ be given. Choose, by Theorem 1.27, an N so that $N > C^2$. If $n \geq N$, then

$$\sqrt{n} \geq \sqrt{N} > C.$$

Example 2.24. Show the sequence $(a_n = \frac{n^2-1}{n+2})$ diverges to ∞.

Observe, for $n \geq 2$, that $n^2 - 1 \geq \frac{1}{2}n^2$ and at the same time $n + 2 \leq 2n$. Thus, for $n \geq 2$,

$$\frac{n^2 - 1}{n + 2} \geq \frac{1}{4}\frac{n^2}{n} = \frac{1}{4n}.$$

Given $C > 0$ choose N such that $N \geq \max\{2, \frac{1}{4C}\}$. With this choice of N, if $n \geq N$, then

$$\frac{n^2 - 1}{n + 2} \geq \frac{1}{4n} \geq \frac{1}{4N} > C.$$

Theorem 2.25. *An increasing sequence (a_n) converges if and only if it is bounded above.*

If (a_n) is increasing, but not bounded above (equivalently diverge), then (a_n) diverges to ∞.

Thus, if (a_n) is increasing, then either (a_n) converges or diverges to ∞ depending on whether it is bounded above or not.

Proof. That a bounded increasing sequence converges has already been established. On the other hand, convergent sequences are bounded. Thus, assuming (a_n) is increasing, convergence and boundedness are equivalent.

To prove the second statement, suppose (a_n) is not bounded and let $C > 0$ be given. Since (a_n) is not bounded, there is an N so that $a_N > C$. Now, if $n \geq N$, then $a_n \geq a_N > C$ and so (a_n) diverges to ∞. □

2.4. **Limit Theorems.**

Theorem 2.26. *Let (a_n) and (b_n) be sequences from $\mathbb{R}$ which converges to A and B respectively.*

(a) The sequence $(a_n + b_n)$ converges to $A + B$;
(b) The sequence $(a_n b_n)$ converges to AB;
(c) If $b_n \neq 0$ for all n and $B \neq 0$, then $(\frac{1}{b_n})$ converges to $\frac{1}{B}$; and
(d) if there is a K so that $a_n \leq b_n$ for all $n \geq K$, then $A \leq B$.

Proof. Item (a) is left as an exercise.

To prove item (b) first observe that (a_n) is a a bounded sequence (since it converges) and hence there is a $C > 0$ such that $|a_n| \leq C$ for all n. Now, given $\epsilon > 0$ there is an N_a such that if $n \geq N_a$, then

$$|a_n - A| < \frac{\epsilon}{2(|B| + 1)}.$$

Similarly, there is a N_b such that, for $n \geq N_b$,

$$|b_n - B| < \frac{\epsilon}{2C}.$$

Let $N = \max N_a, N_b$. If $n \geq N$, then

$$\begin{aligned} |a_n b_n - AB| \leq & |a_n(b_n - B)| + |(a_n - A)B| \\ \leq & C|b_n - B| + |B|\,|a_n - A| \\ \leq & \frac{\epsilon}{2} + \frac{\epsilon}{2}. \end{aligned}$$

To prove item (c), first note that, with $\epsilon = \frac{|B|}{2}$, there is a K so that $|b_n - B| \leq \frac{|B|}{2}$ for $n \geq K$. Thus, $|b_n| \geq \frac{|B|}{2}$ for $n \geq K$. Now, given $\epsilon > 0$ there is an M such that for $n \geq N$,

$$|b_n - B| < \frac{\epsilon |B|^2}{2}.$$

Choose $N = \max\{K, M\}$. If $n \geq N$, then

$$|\frac{1}{b_n} - \frac{1}{B}| = \frac{|b_n - B|}{|b_n B|} \leq \frac{2|b_n - B|}{|B|^2} < \epsilon.$$

To prove item (iv), let $\epsilon > 0$ be given. There exists N_a and N_b so that $a_n > A - \epsilon$ and $b_n < B - \epsilon$ for $n \geq N_a$ and $n \geq N_b$ respectively. Hence, for any $m \geq \max\{N_a, N_b\}$

$$A - \epsilon < a_m \leq b_m < B + \epsilon.$$

Hence $A - B < 2\epsilon$. Since $\epsilon > 0$ is arbitrary, it follows that $A - B \leq 0$. □

Proposition 2.27. *Suppose (a_n) is a sequence of non-negative numbers and $q \in \mathbb{Q}^+$ (q is a positive rational number). If (a_n) converges to A, then (a_n^q) converges to A^q.*

The proof exploits the identity, for $m \in \mathbb{N}^+$,

$$(y-x)\sum_{j=0}^{m-1} y^j x^{m-j-1} = y^m - x^m.$$

Proof. Fix $m \in \mathbb{N}^+$. Since (a_n) converges, there is a $C \geq 1$ such that $C \geq A$ and $C \geq a_n$ for all n. In particular,

$$\sum_{j=0}^{m-1} a_n^j A^{m-1-j} \leq mC^{\frac{m-1}{m}} \leq mC.$$

Given $\epsilon > 0$, there is an N such that if $n \geq N$, then

$$|A - a_n| \leq \frac{\epsilon}{mC}.$$

Hence, for such n,

$$|A^m - a_n^m| = |A - a_n| \sum_{j=0}^{m-1} a_n^j A^{m-j-1} \leq \frac{\epsilon}{mC} mC = \epsilon.$$

Suppose $A = 0$; i.e., (a_n) converges to 0. In this case, to see $(a_n^{\frac{1}{m}})$ converges to 0, given $\epsilon > 0$, note that there is an N such that

$$0 \leq a_n < \epsilon^m.$$

Thus, for $n \geq N$,

$$|a_n^{\frac{1}{m}}| \leq \epsilon.$$

Now suppose $A > 0$. Replace y by $A^{\frac{1}{m}}$ and x by $a_n^{\frac{1}{m}}$ gives,

$$|A^{\frac{1}{m}} - a_n^{\frac{1}{m}}| = \frac{|A - a_n|}{\sum_{j=0}^{m-1} A^{\frac{j}{m}} a_n^{\frac{m-j-1}{m}}} \leq |A - a_n| \frac{1}{A^{\frac{1}{m}}}.$$

From here it is easy to show that $(a_n^{\frac{1}{m}})$ converges to $A^{\frac{1}{m}}$. Finally, given the rational number $q = \frac{m}{k}$, note that from what has already been proved that $b_n = a_n^m$ converges to $B = A^m$. Thus, again by what has already been proved, $b_n^{\frac{1}{k}}$ converges to $B^{\frac{1}{k}}$ and the proof is complete. □

Example 2.28. Recall that the sequence (a_n) defined recursively in Example 2.24 converges to some A. By Proposition 2.27, $(\sqrt{a_n})$ converges to $\sqrt{A}$. Hence, $(2 + \sqrt{a_n})$ converges to $(2 + A)$ by Theorem 2.26. Another application of Proposition 2.27 implies that

$$\lim \sqrt{2 + \sqrt{a_n}} = \sqrt{2 + \sqrt{A}}.$$

Now, the recursive definition of a_{n+1} and Proposition 2.10, imply

$$A = \sqrt{2 + \sqrt{A}}.$$

Hence $1 \le A \le 2$ is a solution to

$$A^4 - 4A^2 - A + 4 = 0.$$

Proposition 2.29 (Squeeze Theorem)**.** *Let (a_n), (b_n), and (c_n) be given sequences.*

If there is a K so that $a_n \le b_n \le c_n$ for $n \ge K$ and if there is an L so that both (a_n) and (c_n) converge to L, then (b_n) converges to L.

If there is a K so that $a_n \le b_n$ for $n \ge K$ and if (a_n) diverges to ∞, then so does (b_n).

Proof. To prove the first statement of the proposition, let $\epsilon > 0$ be given. There exists N_a and N_c such that $|a_n - L| < \epsilon$ and $|c_n - L| < \epsilon$ for $n \ge N_a$ and $n \ge N_c$ respectively. Let $N = \max\{N_a, N_c, K\}$. For $n \ge N$,

$$-\epsilon < a_n - L \le b_n - L \le c_n - L < \epsilon.$$

Hence $|b_n - L| < \epsilon$ for $n \ge N$ and thus (b_n) converges to L.

The proof of the second statement is left as an exercise. □

Proposition 2.30. *Suppose (a_n) is a sequence of positive numbers and that*

$$R = \lim_n \frac{a_{n+1}}{a_n}$$

exists. If $R < 1$, then the sequence (a_n) converges to 0. If $R > 1$, then the series diverges to ∞.

Proof. First suppose $R < 1$. Choose ρ such that $R < \rho < 1$. There exists an N so that

$$0 < \frac{a_{n+1}}{a_n} < \rho$$

for $n \ge N$. In particular, $a_{N+1} \le \rho a_N$. Iterating this inequality gives $a_{N+2} \le \rho a_{N+1} \le \rho^2 a_N$. By induction, it follows that $0 \le a_{N+n} \le \rho^n a_N$. Since the sequence $(a_N \rho^n)_n$ converges to 0 (as does the sequence (0)), Proposition 2.29 implies that $(a_{N+n})_n$ converges to 0. Hence (a_n) itself converges to 0.

The case of $R > 1$ is left as an exercise. □

Proposition 2.31. *Suppose (a_n) and (b_n) are sequences of positive numbers. If (b_n) converges to $B > 0$ and $\frac{a_n}{b_n}$ converges to L, then (a_n) converges to LB.*

Similarly, if (b_n) diverges to ∞ and if $\frac{a_n}{b_n}$ converges to some $L > 0$, then (a_n) diverges to ∞.

There are other variations of this proposition.

Proof. Let $c_n = \frac{a_n}{b_n}$. The sequences (c_n) and (b_n) converge to B and L respectively. Hence the sequence $(b_n c_n) = (a_n)$ converges to BL.

The second part of the proposition is proved similarly. □

Example 2.32. Show $(a_n = \frac{n^2-1}{n+2})$ diverges to ∞.

Let $b_n = n$. Then,

$$\frac{a_n}{b_n} = \frac{1 - \frac{1}{n^2}}{1 + \frac{2}{n}}$$

which, using various limits theorems, converges to 1. Since (b_n) diverges to ∞, it follows that (a_n) does too.

This section closes with a couple of concrete limits.

Proposition 2.33. *Fix a positive number c. Both sequence $(c^{\frac{1}{n}})$ and $(n^{\frac{1}{n}})$ converge to 1.*

Proof. For a real number x, the Binomial Theorem gives,

$$(1+x)^n = \sum_{j=0}^{n} \binom{n}{j} x^j.$$

For $x > 0$ it follows that

$$(1+x)^n \geq 1 + nx.$$

Thus, if $c > 1$ and $x = c^{\frac{1}{n}} - 1$, then

$$c - 1 \geq nc^{\frac{1}{n}}.$$

Dividing by n and using the fact that $\frac{1}{n}$ converges to 0 and proves $(c^{\frac{1}{n}})$ converges to 0.

If $0 < c < 1$, then $\frac{1}{c} > 0$ and from what has already been proved, $((\frac{1}{c})^{\frac{1}{n}})$ converges to 1. Hence, by Theorem 2.26, $(c^{\frac{1}{n}})$ converges to $\frac{1}{1} = 1$ too.

To prove the second part of the Proposition, note that the Binomial Theorem gives, for $x > 0$,

$$(1+x)^n \geq \frac{n(n-1)}{2} x.$$

Thus, with $x = n^{\frac{1}{n}} - 1$,

$$n \geq \frac{n(n-1)}{2} x.$$

Hence,

$$\frac{2}{n-1} \geq n^{\frac{1}{n}} - 1 \geq 0,$$

from which it follows that $(n^{\frac{1}{n}})$ converges to 1. □

2.5. **Super Cauchy sequences and the contraction principle.**

Definition 2.34. A sequence (a_n) is *super Cauchy* if there is a C such that

$$\sum_{1}^{n} |a_{j+1} - a_j| \leq C$$

for all n.

Lemma 2.35. *If (a_n) is super Cauchy, then (a_n) is Cauchy.*

Proof. The sequence,

$$s_n = \sum_{1}^{n-1} |a_{j+1} - a_j|$$

is increasing and bounded above by C. Hence (s_n) is convergent and therefore Cauchy. In particular, given $\epsilon > 0$ there is an N so that if $n, m \geq N$, then $|s_m - s_n| < \epsilon$. Hence, for $m \geq n \geq N$,

$$|a_m - a_n| \leq \sum_{j=n}^{m-1} |a_{j+1} - a_j| = |s_{m+1} - s_n| < \epsilon.$$

□

Proposition 2.36 (Contraction Principle)**.** *Suppose (a_n) is a sequence from $\mathbb{R}$. If there is an N and an $0 \leq r < 1$ such that*

$$|a_{n+2} - a_{n+1}| \leq r|a_{n+1} - a_n| \tag{1}$$

for all $n \geq N$, then (a_n) is super Cauchy and hence converges.

Proof. It can be assumed that equation (1) holds for all n. In that case, an induction argument shows,

$$|a_{j+1} - a_j| \leq r^j |a_1 - a_0|$$

Summing over j gives,

$$\sum_{1}^{n} |a_{j+1} - a_j| \leq |a_1 - a_0| \sum_{1}^{n} r^j.$$

On the other hand,

$$\sum_{1}^{n} r^j = r\frac{1 - r^n}{1 - r} \leq \frac{r}{1 - r}.$$

□

Example 2.37.

2.6. **Subsequences.**

Definition 2.38. Suppose (a_n) is a sequence from $\mathbb{R}$. If $n_1 < n_2 < \cdots$ is an increasing sequence of integers, then the sequence $(a_{n_j})_j$ is *subsequence* of (a_n).

Example 2.39. Given $(a_n = (-1)^m)$ both $(b_j = a_{2j} = 1)$ and $(c_j = a_{2j+1} = -1)$ are subsequences of (a_n).

Similarly, choosing $n_j = j^2$, the sequence $(\frac{1}{2^j})$ is a subsequence of $(\frac{1}{j})$.

Definition 2.40. A point A is a *subsequential limit* of the sequence (a_n) if there is a subsequence (a_{n_j}) of (a_n) which converges to A.

Example 2.41. The points 1 and -1 are both subsequential limits of the sequence $((-1)^n)$.

Lemma 2.42. *Let (a_n) be a given sequence. If A is an accumulation point of $S = \{a_n : n \in \mathbb{N}\}$, then A is a subsequential limit of the sequence (a_n).*

Proof. The set $(A-1, A+1) \cap S$ is infinite. Hence, there exists an n_1 such that $|a_{n_1} - A| < 1$. The set $S \cap (A - \frac{1}{2}, A + \frac{1}{2})$ is infinite. Hence, there is an $n_2 > n_1$ such that $|a_{n_2} - A| < \frac{1}{2}$. Continuing in this fashion (recursively), constructs $n_1 < n_2 < n_3 < \dots$ such that $|a_{n_j} - A| < \frac{1}{j}$. Thus (a_{n_j}) is a subsequence of (a_n) which converges to A. □

Theorem 2.43. *A bounded sequence has a convergent subsequence.*

Proof. Suppose (a_n) is a bounded sequence. Thus, there is a C such that $|a_n| \leq C$ for all n. Let $S = \{a_n : n\}$ denote the range of the sequence. Suppose S is infinite. In this case S has an accumulation point A by Theorem 1.34. By Lemma 2.42, a subsequence of (a_n) converges to A; i.e., (a_n) has a convergent subsequence.

If S is finite, then there is an A such that $A = a_n$ for infinitely many n. It is an easy exercise, left to the reader, to show that there is a subsequence of (a_n) which converges to A. □

Proposition 2.44. *If (a_n) converges to A and (a_{n_j}) is a subsequence, then (a_{n_j}) converges to A.*

Example 2.45. The sequence $((-1)^n)$ diverges.

2.6.1. *The limits superior and inferior.*

Definition 2.46. Suppose (a_n) is a bounded sequence. Let, for $m \in \mathbb{N}$,

$$b_m = \sup\{a_n : n \geq m\}.$$

The sequence (b_m) is decreasing and bounded below (by any lower bound for (a_n)). Hence (b_m) converges to some L which is called the *limit superior* or *limsup* of (a_n) and is denoted by $\limsup a_n$ or $\overline{\lim} a_n$.

The *liminf*, denoted $\underline{\lim}$ or $\liminf$ is defined analogously.

Example 2.47. Find the limsup of the sequence $(a_n = \sin(n\frac{\pi}{2}))$.

First observe that the range of the sequence is the bounded set $S = \{0, 1, -1\}$. Hence the sequence has both a limsup and a liminf. Further Given an m, the set $\{a_n : n \geq m\} = S$. Hence, in the notation above, $b_m = 1$ for all m. It follows that $\limsup a_n = \lim b_m = 1$. Similarly, $\liminf a_n = -1$.

The proofs Propositions 2.48 and 2.49 below are left to the interested reader.

Proposition 2.48. *A sequence (a_n) converges if and only if it is bounded and $\limsup a_n = \liminf a_n$.*

The following proposition says that $\limsup a_n$ is the largest subsequential limit of the sequence (a_n) (and in particular asserts that a largest exists). This gives another rationale for the name limsup.

Proposition 2.49. *Suppose (a_n) is bounded. There is a subsequence (a_{n_j}) of (a_n) which converges to $\limsup a_n$. Moreover, if (a_{n_k}) is any convergent subsequence, then $\lim_k a_{n_k} \le \limsup a_n$.*

Example 2.50. Find the limsup of the sequence $a_n = (-1)^n(1+\frac{1}{n})$.

Let $c_n = 1 + \frac{1}{n}$ and observe that c_n converges to 1. Suppose A is a subsequential limit of (a_n). Hence, there is a subsequence (a_{n_j}) of (a_n) which converges to A. In this case $a_{n_j} \le |a_{n_j}| = (1 + \frac{1}{n_j}) = c_{n_j}$. It follows that $A = \lim_j a_{n_j} \le \lim_j c_{n_j} = 1$. On the other hand, the subsequence $a_{2n} = 1 + \frac{1}{2n}$ of (a_n) converges to 1. Thus, 1 is a subsequential limit of (a_n). Hence $1 = \limsup a_n$.

2.7. Problems.

Problem 2.1. Let $a_n = \frac{n-2}{2n+3}$. Show, directly from the definition of limit, that (a_n) converges to $\frac{1}{2}$.

Problem 2.2. Let $a_n = \frac{2n^2-n+1}{n^2+n+3}$. Show (a_n) converges.

Problem 2.3. Let $b_n = \frac{n+3}{n^2+n+3}$. Show (b_n) converges to 0.

Problem 2.4. For $n \ge 2$, let $b_n = \frac{n+3}{n^2-n-1}$. Prove, directly from the definition of limit, that (b_n) converges.

Problem 2.5. In Problem 2.4, rewrite

$$b_n = \frac{\frac{1}{n} - \frac{3}{n^2}}{1 - \frac{1}{n} - \frac{1}{n^2}}$$

and use both known limits and limit theorems to show (b_n) converges to 0. Carry out a similar program with Problem 2.1

Problem 2.6. Let $a_0 = 1$ and define, recursively, $a_{n+1} = \sqrt{2+a+n}$. Prove, by induction, that $a_n \le 2$ for all n and that (a_n) is increasing. Conclude that (a_n) converges. Identify the limit.

Problem 2.7. Fix $r > 1$. Let $a_1 = 1$ and define recursively,

$$a_{n+1} = \frac{1}{r}(a_n + r + 1).$$

Show, that (a_n) is increasing. Show by induction that (a_n) is bounded above by $\frac{r+1}{r-1}$. Does the sequence converge? If so, identify the limit.

Problem 2.8. Fix $a > 1$. Show that the sequence (a^n) diverges to ∞. (Suggestion: It is not too hard to show that (a^n) is increasing, but not Cauchy.)

Problem 2.9. Let $a_n = \sin(\frac{n\pi}{4})$. Show (a_n) is not Cauchy. Conclude that (a_n) doesn't converge.

Problem 2.10. Show that a is an accumulation point of a set D if and only if there is a sequence (a_n) from $D \setminus \{a\}$ which converges to a. Perhaps this result explains the reason that *limit point* is a synonym for accumulation point.

Problem 2.11. Let $F_0 = 0$ and $F_1 = 1$ and define, recursively,

$$F_{n+1} = F_n + F_{n-1}$$

(the Fibonacci sequence). Let $a_n = \frac{F_{n+1}}{F_n}$. Is the sequence (a_n) monotone?

Show $a_{n+1}a_n \geq 2$. Show,

$$|a_{n+1} - a_n| = \frac{|a_{n-1} - a_n|}{a_n a_{n-1}}.$$

Conclude that (a_n) converges. Identify the limit.

Problem 2.12. Let $a_0 = 1$ and define, recursively,

$$a_{n+1} = \frac{a_n + \frac{2}{a_n}}{2}.$$

Is the sequence (a_n) monotone?

Show, by induction, that

(i) $a_n^2 \geq 1$ for all n;
(ii) $a_n^2 \leq 3$; and
(iii) $\frac{5}{2} \geq a_{n+1}a_n \geq \frac{3}{2}$.

Conclude that

$$|\frac{1}{2} - \frac{1}{a_{n+1}a_n}| \leq \frac{1}{2}.$$

Show

$$|a_{n+1} - a_n| \leq \frac{1}{2}|a_n - a_{n-1}|.$$

Explain why (a_n) converges and find, if possible, its limit.

3. Functions and Limits

3.1. **Definitions and examples.**

Definition 3.1. A function f consists of sets A and B, called the *domain* and *codomain* of f respectively, and a rule which assigns to each $a \in A$ a unique $b = f(a) \in B$. We write, $f : A \to B$.

The function f is *one-one* if $f(x) = f(y)$ implies $x = y$; and f is *onto* if $\{f(x) : x \in A\} = B$.

In the case that B is a subset of $\mathbb{R}$ we say that f is *real-valued.*

For the most part, A will be a subset of $\mathbb{R}$ and often $B = \mathbb{R}$.

Example 3.2. Here are a couple examples of functions.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Here both the domain and codomain of f is $\mathbb{R}$.

(b) Define $g : [0,\infty) \to \mathbb{R}$ by $g(x) = x^2$. Note, g is one-one, whereas f above is not.
(c) Define $h : [0,1] \to [0,1]$ by $h(x) = 1$ if $x \in \mathbb{Q} \cap [0,1]$ and $h(x) = 0$ otherwise.

Definition 3.3. Suppose $D \subset \mathbb{R}$, the real number a is an accumulation point of D, and $f : D \to \mathbb{R}$. We say that f *has a limit at* a if there exists a real number L such that for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. In this case we write,

$$L = \lim_{x\to a} f(x)$$

and call L the *limit of* f *at* a.

Example 3.4. Let $D = \mathbb{R}$ and $f(x) = x^2$. Show $\lim_{x\to 1} f(x) = 1$.

Given $\epsilon > 0$ choose $\delta = \min\{1, \frac{\epsilon}{3}\}$. Then, if $|x - 1| < \delta$, then

$$|f(x) - 1| = |x + 1|\,|x - 1| \le 3|x - 1| < \epsilon.$$

Example 3.5. Let $D = (-\infty, 1) \cup (1, \infty) = \mathbb{R} \setminus \{1\}$ and define $g : D \to \mathbb{R}$ by $g(x) = x^2$. Show, $\lim_{x\to 1} g(x) = 1$.

Let $D = \mathbb{R}$ and define $h : D \to \mathbb{R}$ by $h(x) = x^2$ for $x \ne 1$ and $h(1) = 0$. Show, $\lim_{x\to 1} h(x) = 1$.

Simply notice that the limit doesn't depend on f being defined at 1 and if it is defined at 1, it doesn't depend upon the value of f at 1. Hence these examples are essentially the same as the previous example

Definition 3.6. Suppose $f : D \to \mathbb{R}$ and $E \subset D$. The function $f|_E : E \to \mathbb{R}$ defined by $f|_E(x) = f(x)$ (for $x \in E$) is the *restriction of* f to E.

Example 3.7. Let $D = \mathbb{R} \setminus \{0\}$ and define $g : D \to \mathbb{R}$ by $g(x) = \frac{x}{|x|}$. Show, $\lim_{x\to\infty} g(x)$ doesn't exist.

Let L be given. First, suppose $L < 0$. Choose $\epsilon_0 = 1$. Given $\delta > 0$, choose $x = \frac{\delta}{2}$. Then $x \in D$ and $0 < |x - 0| < \delta$, but $|f(x) - 0| = 1 \ge \epsilon_0$. Thus, $L \ne \lim_{x\to 0} g(x)$.

A similar argument shows if $L \ge 0$, then $\lim_{x\to 0} g(x) \ne L$. Hence g does not have a limit at 0.

Example 3.8. Define $f : (0,\infty) \to \mathbb{R}$ by $f(x) = \frac{x}{|x|}$. Thus, f is the restriction of g from Example 3.7 to the set $E = (0,\infty) \subset D$. Show, $\lim_{x\to 0} f(x) = 1$.

Example 3.9. Define $F : (0,\infty) \setminus \{1\} \to \mathbb{R}$ by $F(x) = \frac{1-x}{1-\sqrt{x}}$. Show F has a limit at 1.

First, observe that

$$\begin{aligned}|\frac{1-x}{1-\sqrt{x}}-2| =&|1+\sqrt{x}-2|\\ =&|1-\sqrt{x}|\\ =&|\frac{1-x}{1+\sqrt{x}}|\\ \le&|1-x|.\end{aligned}$$

Thus, given $\epsilon > 0$, if we choose $\delta = \epsilon$, then if $x \in D$ and $|x-1| < \delta$, then $|f(x)-2| < \epsilon$. Hence f has a limit at 1 and this limit is 2.

These examples illustrate that the definition of limit does not require a to be in the domain of f (it just needs to be a limit point of the domain), nor does it depend upon the value of f at a if a is in the domain of f. They also demonstrate, that the limit depends crucially on the domain.

Example 3.10. Define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q} \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q > 0 \text{ and } \gcd(p,q) = 1. \end{cases}$$

Show, for each $a \in (0,1)$, that $\lim_{x\to a} f(x) = 0$.

Let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}^+$ so that $\frac{1}{N} < \epsilon$. Consider the set

$$S_N = \{\frac{m}{n} : m, n \in \mathbb{N}^+, \quad n \le N, \quad m \le n\}.$$

Since S_N is finite,

$$\delta = \min\{|s-a| : s \in S, \quad s \neq a\} > 0.$$

In particular, $(a-\delta, a+\delta) \cap S_N \subset \{a\}$. Hence, if $0 < |a-x| < \delta$, then either $x \notin \mathbb{Q}$ in which case $|f(x)-0| = |0-0| < \epsilon$; or $x \in \mathbb{Q}$ and $x = \frac{p}{q}$ where $q > N$, in which case $|f(x)-0| = |\frac{1}{q}| < \frac{1}{N} < \epsilon$.

Proposition 3.11. *Suppose $f : D \to \mathbb{R}$ and a is an accumulation point of D. If f has a limit as x approaches a, then for every $\epsilon > 0$ there is an $\eta > 0$ such that if $x, y \in D$ and both $|x-a|, |y-a| < \eta$, then $|f(x)-f(y)| < \epsilon$.*

The proof of the proposition is left as an exercise.

Example 3.12. Define $f : (0,\infty) \to \mathbb{R}$ by

$$f(x) = \sin(\frac{1}{x}).$$

Show f does not have a limit at 0.

Observe that 0 is in fact an accumulation point of $D = (0,\infty)$, the domain of f. With $\epsilon_0 = 1$, given $\eta > 0$ there exists $n \in \mathbb{N}^+$ such that $\frac{1}{2n\pi} < \eta$. With $x = \frac{1}{2n\pi}$ and $y = \frac{1}{(2n+\frac{1}{2})\pi}$, we have $0 < x, y < \eta$, but $|f(x)-f(y)| = 1 \ge \epsilon_0$. Hence, by the (contrapositive of the) proposition, f does not have a limit at 0.

3.2. **The sequential formulation of limit of a function.**

Proposition 3.13. *Suppose $f : D \to \mathbb{R}$ and a is an accumulation point of D.*

If $\lim_{x\to a} f(x)$ exists and equals L and if (a_n) is a sequence from $D \setminus \{a\}$ which converges to a, then $\lim n \to \infty f(a_n)$ exists and equals L.

Conversely, if there is an L such that for any sequence (a_n) from $D \setminus \{a\}$ which converges to L the limit $\lim_{n\to\infty} f(a_n) = L$, then $\lim_{x\to a} f(x) = L$.

Proof. First suppose $\lim_{x\to a} f(x) = L$ and that (a_n) is a sequence from $D \setminus \{a\}$ which converges to a. To prove $(f(a_n))$ converges to L, let $\epsilon > 0$ be given. There is a $\delta > 0$ such that if $0 < |x - a| < \delta$ and $x \in D$, then $|f(x) - L| < \epsilon$. There is an N so that if $n \geq N$, then $0 < |a_n - a| < \delta$. Hence, if $n \geq N$, then $|f(a_n) - L| < \epsilon$.

To prove the second statement, suppose $\lim_{x\to a} f(x) \neq L$. Thus, there is an $\epsilon_0 > 0$ such that for every $\delta > 0$ there is a point $x \in D$ such that $0 < |x - a| < \delta$, but $|f(x) - L| \geq \epsilon_0$.

Thus, with $\delta_n = \frac{1}{n}$, there exists $a_n \in D$ such that $0 < |a_n - a| < \delta_n$ and $|f(a_n) - L| \geq \epsilon_0$. It follows that (a_n) converges to a, but $(f(a_n))$ does not converge to L. □

Corollary 3.14. *Suppose $f : D \to \mathbb{R}$ and a is an accumulation point of D. If there exists a sequence (a_n) from $D \setminus \{a\}$ such that (a_n) converges to A, but $(f(a_n))$ diverges, then f does not have a limit as x tends to a.*

Similarly, if there exists sequences (a_n) and (b_n) from $D \setminus \{a\}$ which converge to a, but $(f(a_n))$ and $f((b_n))$ don't converge to the same value (which includes the case that one or both diverges), then f does not have a limit as x approaches a.

Example 3.15. Let $D = \mathbb{R} \setminus \{0\}$ and define $f : D \to \mathbb{R}$ by $f(x) = \sin(\frac{1}{x})$. Show that f does not have a limit at 0.

Choose $a_n = \frac{1}{(n+\frac{1}{2})\pi}$ for $n \in \mathbb{N}$. Note that (a_n) converges to 0, but $f(a_n) = (-1)^n$ which diverges.

Example 3.16. Let $f : [0,1] \to \mathbb{R}$ denote the *indicator function* (synonymously *characteristic function*) of $\mathbb{Q} \cap [0,1]$ defined by

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

Show, for each $a \in (0,1)$ that $\lim_{x\to a} f(x)$ does not exists.

Fix a. Since a is a limit point of $\mathbb{Q}$, there is a sequence (a_n) from $\mathbb{Q} \cap (0,1)$ which converges to a and such that for each n, $a_n \neq a$. Since $a_n \in \mathbb{Q}$, we have $(f(a_n)) = (1)$ converges to 1. There is also a sequence $b_n from [0,1] \setminus \mathbb{Q}$ such that (b_n) converges a and for each n, $b_n \neq a$. We have $(f(b_n)) = (0)$ converges to 0. Thus, f does not have a limit at a.

3.3. **Infinite limits and limits at infinity.**

Definition 3.17. Suppose $f : D \to \mathbb{R}$ and a is an accumulation point of D. The *limit of f as x approaches a is ∞* if for every $C > 0$ there is a δ such that if $x \in D$ and $0 < |x - a| < \delta$, then $f(x) > C$, denoted

$$\lim_{x \to a} f(x) = \infty.$$

Suppose $f : D \to \mathbb{R}$ and for every $C > 0$ there is an $x \in D$ such that $x > D$ and that $A \in \mathbb{R}$. The *limit of f as x approaches ∞ is A* if for every $\epsilon > 0$ there is a $C > 0$ such that if $x \in D$ and $x > C$, then $|f(x) - A| < \epsilon$, denoted

$$\lim_{x \to \infty} f(x) = A.$$

The expression,

$$\lim_{x \to \infty} f(x) = \infty$$

is defined similarly.

Example 3.18. Show,

$$\lim_{x \to \infty} \frac{1}{x^2} = 0.$$

Implicitly, $D = \mathbb{R} \setminus \{0\}$ and $f(x) = x^{-2}$. Given $\epsilon > 0$ choose $C = \epsilon^{-\frac{1}{2}}$. Now, if $x > C$, then $0 < x^{-2} - 0 < C^{-2} = \epsilon$.

Example 3.19. Let $D = (0, \infty)$ and define $g : D \to \mathbb{R}$ by $g(x) = x^{-1}$. Show

$$\lim_{x \to 0} g(x) = \infty.$$

Note, this is sometimes expressed as

$$\lim_{x \to 0+} \frac{1}{x} = \infty$$

Given $C > 0$, choose $\epsilon = \frac{1}{C} > 0$. If $x \in D$ and $|x - 0| < \epsilon$, then $0 < x < \frac{1}{C}$ and hence $0 < g(x) = x^{-1} > C$.

Example 3.20. Assuming knowledge of the log function, show,

$$\lim_{x \to \infty} \log(x) = \infty.$$

Recall, $\log(2^k) = k \log(2)$ for $k \in \mathbb{N}$ and $\log(2) > 0$. In particular, the sequence $(\log(2^k))$ diverges to ∞. Moreover, if $y > x > 0$, then $\log(y) > \log(x)$. Given $K > 0$ choose k such that $2^k \log(2) > K$. Choose $C = 2^k$. If $x > C$, then $\log(x) > \log(C) = k \log(2) > K$.

Proposition 3.21. *Suppose $D \subset (0, \infty)$. If for every $C > 0$ there exists an $x \in D$ such that $x > C$, then 0 is a limit point of $E = \{\frac{1}{x} : x \in D\}$.*

Suppose $f : D \to \mathbb{R}$ and define $g : E \to \mathbb{R}$ by $g(x) = f(\frac{1}{x})$. Then

$$\lim_{x \to \infty} f(x) = \lim_{x \to 0} g(x).$$

Remark 3.22. The conclusion is that either both limits exist and are equal or they both fail to exist. The proof is left as an exercise.

Example 3.23. Redo Example 3.18. Let $D = (0, \infty)$ and define $g : D \to \mathbb{R}$ by $g(x) = x^2$. To see

$$\lim_{x \to 0} g(x) = 0,$$

let $\epsilon > 0$ be given. Choose $\delta = \min\{1, \epsilon\}$. Now, if $x \in D$ and $|x - 0| < \delta$, then

$$|g(x) - 0| = x^2 < x < \delta \leq \epsilon,$$

where we have used $0 < x < 1$ in the first inequality.

Observe that a similar argument shows $\lim_{x\to\infty} \frac{1}{x} = 0$ too.

There are sequential formulations of limits at infinity and infinite limits. Rather than state all the variations, we offer the following proposition as a sample result.

Proposition 3.24. *Suppose $f : D \to \mathbb{R}$ and a is an accumulation point of D. If $\lim_{x\to a} f(x) = \infty$ and if (a_n) is a sequence from $D \setminus \{a\}$ which converges to a, then $\lim_{n\to\infty} f(a_n) = \infty$.*

Example 3.25. Show $\lim_{x\to 0} \frac{1}{x}$ does not exists and is not either $\pm\infty$. Implicitly, here we have $D = \mathbb{R} \setminus \{0\}$ and $f(x) = \frac{1}{x}$.

Consider the sequence $(a_n = \frac{1}{n})_{n=1}^{\infty}$. The sequence $(b_n = f(a_n)) = (n)$ diverges to ∞. Hence, no real number or $-\infty$ can be the limit. On the other hand, with $(c_n = -a_n)$ the sequence $(f(c_n)) = (-n)$ diverges to $-\infty$ and thus ∞ can not be the limit. Hence the limit fails to exist even in the sense of $\pm\infty$.

This section closes with two simple examples.

Example 3.26. Suppose k is a real number and $f : D \to \mathbb{R}$ is the constant function $f(x) = k$. If a is an accumulation point of D, then $\lim_{x\to a} f(x) = k$.

Example 3.27. If $f : D \to \mathbb{R}$ is the identity function, $f(x) = x$, and if a is an accumulation point of D, then $\lim_{x\to a} f(x) = a$.

3.4. **Limit Theorems.** The following theorem says, like in the case of limits of sequences, limits of functions are compatible with the algebraic operations on $\mathbb{R}$.

Proposition 3.28. *Suppose $f, g, h : D \to \mathbb{R}$ and a is an accumulation point of D. If the limits $\lim_{x\to a} f(x)$, $\lim_{x\to a} g(x)$ and $\lim_{x\to a} h(x)$ exist and equal A, B, C respectively, then*

(i) $\lim_{x\to a}(f + g)(x) = A + B$;
(ii) $\lim_{x\to a} fg(x) = AB$; *and*
(iii) *if $C \neq 0$ and h is never 0, then* $\lim_{x\to a} \frac{1}{h(x)} = \frac{1}{C}$.

Proof. Suppose (a_n) is a sequence from $D \setminus \{a\}$ which converges to a. By Proposition 3.13, $\lim_{n\to\infty} f(a_n) = A$ and similarly, $\lim_{n\to\infty} g(a_n) = B$. It follows, from Theorem 2.26, that both $\lim_{n\to a} fg(a_n) = AB$ and $\lim_{n\to\infty}(f+g)(a_n) = A+B$. Hence another application of Proposition 3.13 proves both items (i) and (ii).

The proof of item (iii) is similar. The details are left as an exercise. □

Remark 3.29. Analogous results hold for limits at infinity and infinite limits, with, in the latter case, the obvious caveats.

Example 3.30. Find

$$\lim_{x\to\infty} \frac{2x-1}{x+3},$$

if it exists.

We interpret the limit as $\lim_{x\to\infty} f(x)$ where $D = \mathbb{R}\setminus\{-3\}$ and $f : D \to \mathbb{R}$ is defined by

$$f(x) = \frac{2x-1}{x+3}.$$

Rewrite f as

$$f(x) = \frac{2-\frac{1}{x}}{1+\frac{3}{x}}.$$

As an exercise, show that $\lim_{x\to\infty} \frac{1}{x} = 0$. Using this fact, $\lim_{x\to\infty} \frac{3}{x} = 0$ and hence,

$$\lim_{x\to\infty} f(x) = \frac{2-\lim_{x\to\infty}\frac{1}{x}}{1+\lim_{x\to\infty}\frac{3}{x}} = \frac{2-0}{1+0} = 2.$$

Remark 3.31. From Examples 3.26 and 3.27 and Proposition 3.28, it follows that if p is a polynomial, then, for any $a \in \mathbb{R}$ that

$$\lim_{x\to a} p(x) = p(a).$$

For instance, choosing $f = g$ to be the function in Example 3.27 and part (ii) of the proposition,

$$\lim_{x\to a} x^2 = a^2.$$

Proposition 3.32. *Suppose $f : D \to [0,\infty)$, a is an accumulation point of D and $q \in \mathbb{Q}^+$. If $\lim_{x\to a} f(x)$ exists an equals L, then,*

$$\lim_{x\to a} f(x)^q = L^q.$$

Example 3.33. Find

$$\lim_{x\to 4} \frac{\sqrt{x}+1}{\sqrt{x+1}}$$

Example 3.34. Find,

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}.$$

With $D = (0, \infty)$ and $f : D \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{\sqrt{x^2+1}},$$

we will show $\lim_{x\to\infty} f(x) = 1$. First, rewrite f as

$$f(x) = \frac{1}{\sqrt{1+\frac{1}{x^2}}}.$$

Now, by Example 3.23, $\lim_{x\to\infty} \frac{1}{x^2} = 0$ and thus $\lim_{x\to\infty} 1 + \frac{1}{x^2} = 1$. It follows that

$$\lim_{x\to\infty} \sqrt{1+\frac{1}{x^2}} = 1$$

and thus,

$$\lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x^2}}} = \frac{1}{\lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}}} = \frac{1}{1} = 1.$$

3.4.1. *The squeeze theorem.*

Proposition 3.35. *Suppose $f, g : D \to \mathbb{R}$ and a is a limit point of D. If there is an $\eta > 0$ such that if $x \in D$ and $0 < |x-a| < \eta$, then $f(x) \le g(x)$ and if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists, then*

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Proof. For notational ease, let $A = \lim_{x\to a} f(x)$ and $B = \lim_{x\to a} g(x)$. Since a is a limit point of D, there exists a sequence (a_n) from D such that $0 < |a_n - a| < \eta$ for all n and (a_n) converges to a. It follows that $f(a_n) \le g(a_n)$ and the sequences $(f(a_n))$ and $(g(a_n))$ converge to A and B respectively. Thus, by Theorem 2.26, $A \le B$. □

Proposition 3.36. *Suppose $f, g, h : D \to \mathbb{R}$ and a is an accumulation point of D. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} h(x)$ exist and equal L and if there is an $\eta > 0$ such that $f(x) \le g(x) \le h(x)$ for $x \in D$ and $0 < |x-a| < \eta$, then $\lim_{x\to a} g(x) = L$.*

Proof. Let $\epsilon > 0$ be given. There exists a $0 < \delta < \eta$ such that if $x \in D$ and $0 < |x-a| < \delta$, then $L - f(x) < \epsilon$ and $h(x) - L < \epsilon$. Hence, for such x,

$$-\epsilon < f(x) - L \le g(x) - L \le h(x) - L < \epsilon$$

and the conclusion follows. □

Example 3.37. Show,

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1.$$

Here choose $D = (0, \infty)$ and define $g : (0, \infty) \to \mathbb{R}$ by

$$g(x) = \frac{x}{\sqrt{x^2+1}}.$$

Define $f, h : (0, \infty) \to \mathbb{R}$ by $h(x) = 1$ and

$$f(x) = 1 - \frac{1}{x}.$$

Verify that the inequalities $f(x) \le g(x) \le h(x)$ hold for $x > 1$. Since both f and h approach 1 as x tends to ∞,

$$\lim_{x \to \infty} g(x) = 1$$

too.

3.4.2. *Compositions.*

Proposition 3.38. *Suppose $g : D \to \mathbb{R}$, the point a is an accumulation point of D, $\lim_{x \to a} g(x)$ exists, is equal A and $A \notin E$ is and accumulation point of E. If $f : E \to \mathbb{R}$ has the limit L at A, then $f \circ g : D \to \mathbb{R}$ has limit L at a.*

Proof. Let $\epsilon > 0$ be given. There is a $\delta > 0$ such that if $0 < |y - A| < \delta$, and $y \in E$, then $|f(y) - L| < \epsilon$. With this $\delta > 0$ there is an $\eta > 0$ such that if $0 < |x - a| < \eta$ and $x \in D$, then $|g(x) - A| < \delta$. Hence, if $0 < |x - a| < \eta$ and $x \in D$, then $g(x) \in E$ so that $0 < |g(x) - A| < \delta$ and therefore, $|f(g(x)) - L| < \epsilon$. □

Example 3.39. Find

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(5x)}.$$

We will assume the fact that, with $D = \mathbb{R} \setminus \{0\}$ and $f : D \to \mathbb{R}$ defined by $f(x) = \frac{\sin(x)}{x}$, we have

$$\lim_{x \to 0} f(x) = 1.$$

Let $g(x) = 4x$ defined on $\mathbb{R} \setminus \{0\}$. Thus g maps into the domain D of f and the hypotheses of the Proposition 3.38 are satisfied. Hence,

$$\lim_{x \to 0} f(g(x)) = \lim_{t \to 0} f(t) = 1.$$

Thus,

$$\lim_{x \to 0} \frac{\sin(4x)}{4x} = 1.$$

Similarly,

$$\lim_{x \to 0} \frac{\sin(5x)}{5x} = 1.$$

Finally,

$$\frac{\sin(4x)}{\sin(5x)} = \frac{4}{5} \frac{\frac{\sin(4x)}{4x}}{\frac{\sin(5x)}{5x}}.$$

Using rules of limits (mostly notably the limit of a quotient is the quotient of the limits provided both limits exist and the limit in the denominator is not zero), we find,

$$\lim_{x \to 0} \frac{\sin(4x)}{\sin(5x)} = \frac{4}{5}.$$

3.5. **One sided limits.**

Definition 3.40. Suppose $D \subset \mathbb{R}$. Given a point $a \in \mathbb{R}$, let

$$D_{a+} = \{x \in D : x > a\}.$$

Given $L \in \mathbb{R}$, if a is a limit point of D_{a+} we say f *has limit L at a from the right (or above)* if

$$\lim_{x \to a} f|_{D_{a+}} = L.$$

In this case we write,

$$L = \lim_{x \to a+} f(x).$$

The notion of the limit of f at a from the left is defined similarly. These limits, to the extent they exist, are *one-sided limits of f at a.*

Example 3.41. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Show both limits $\lim_{x \to 0+} h(x)$ and $\lim_{x \to 0-} h(x)$ exist.

Proposition 3.42. *Suppose $f : D \to \mathbb{R}$ and $a \in \mathbb{R}$. If a is a limit point of both D_{a+} and D_{a-}, then $\lim_{x \to a} f(x)$ exists if and only if both one sided limits at a exists and are equal. In this case,*

$$\lim_{x \to a-} f(x) = \lim_{x \to a} f(x) = \lim_{x \to a+} f(x).$$

Remark 3.43. The proof is left as an easy exercise.

A similar result holds for infinite limits.

Example 3.44. The function h in Example 3.41 does not have a limit at 0.

Example 3.45. Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} x^3 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0. \end{cases}$$

Show g has limit 0 at 0.

Let p denote the function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^2$. Since p is a polynomial, $\lim_{x \to 0} p(x) = p(0) = 0$. Hence, $\lim_{x \to 0+} g(x) = \lim_{x \to 0+} p(x) = 0$. Similarly, $\lim_{x \to 0-} g(x) = \lim_{x \to 0-} x^3 = 0$. Thus $\lim_{x \to 0} g(x) = 0$.

3.6. **Monotone functions.**

Definition 3.46. A function $f : D \to \mathbb{R}$ is *increasing* if $x, y \in D$ and $x < y$ implies $f(x) \leq f(y)$.

Proposition 3.47. *If $f : D \to \mathbb{R}$ is increasing and $a \in D$ is an accumulation point of D_{a-}, then $\lim_{x \to a-} f(x)$ exists.*

Proof. Note that if $x < a$ and $x \in D$, then $f(x) \le f(a)$. Hence the set

$$S = \{f(x) : x \in D, \quad x < a\}$$

is bounded above by $f(a)$. Since a is an accumulation point of D_{a-} the set D_{a-}, and hence S, is nonempty. Therefore S has a least upper bound L. To see that $\lim_{x\to a-} f(x) = L$, let $\epsilon > 0$ be given. By the least property of L, there exists a $z \in D$ with $z < a$ and $f(z) > L - \epsilon$. Let $\delta = a - z$. If $0 < a - x < \delta$ and $x \in D$, then $z < x < a$ so that $f(z) \le f(x) \le f(a)$. Thus $L - \epsilon < f(z) \le f(x) \le L$ so that

$$|f(x) - L| < \epsilon.$$

□

Remark 3.48. By a similar argument, if a is also a limit point of D_{a+}, then $\lim_{x\to a+} f(x)$ exists and moreover,

$$\lim_{x\to a-} f(x) \le f(a) \le \lim_{x\to a+} f(x).$$

Problem 3.1. Suppose $D \subset \mathbb{R}$, for each $K > 0$ there is a point $s \in D$ such that $s > K$. Show, if there is an R such that if $R < x < y$ and $x, y \in D$, then $f(x) \le f(y)$, then $\lim_{x\to\infty} f(x)$ exists or $\lim_{x\to\infty} f(x) = \infty$. Informally, the problem says, if f is *eventually increasing*, then $\lim_{x\to\infty} f(x)$ exists as an *extended real number*.

4. Continuity

Definition 4.1. Given $D \subset \mathbb{R}$, a point $a \in D$, the function $f : D \to \mathbb{R}$ is *continuous at* a if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in D$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

The function f is *continuous* if it is continuous at each point $a \in D$.

Remark 4.2. If a is an accumulation point of D, then f is continuous at a if and only if

$$\lim_{x\to a} f(x) = f(a).$$

If a is not an accumulation point of D, then f is continuous at a. In particular, and as an example, every function $f : \mathbb{Z} \to \mathbb{R}$ is continuous.

Example 4.3. Show that the function h from Example 3.2 is nowhere continuous.

Example 4.4. The function from Example 3.10 is continuous at the irrational points in $(0, 1)$ and discontinuous at the rational points in $(0, 1)$.

Many of our facts about limits can be interpreted in terms of continuity.

Example 4.5. Fix $k \in \mathbb{R}$ and define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = k$. Show that h is continuous.

Fix a point $a \in \mathbb{R}$. By example 3.26, $\lim_{x\to a} h(x) = k$. Thus, $\lim_{x\to a} h(x) = h(a)$ and h is continuous at a. Since a was arbitrary, h is continuous.

Example 4.6. Show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ (the identity function) is continuous.

Fix a point $a \in \mathbb{R}$. By Example 3.27, $\lim_{x\to a} f(x) = a$. Thus, $\lim_{x\to a} f(x) = f(a)$ and f is continuous at a. Hence f is continuous.

Definition 4.7. Given polynomials p and q, let $D = \{x \in \mathbb{R} : q(x) \neq 0\}$. The function $r : D \to \mathbb{R}$ defined by $r(x) = \frac{p(x)}{q(x)}$ is a *rational function*.

Example 4.8. Polynomials and rational functions are continuous (on their domains).

Example 4.9. Given $q \in \mathbb{Q}^+$, then function $s : [0, \infty) \to [0, \infty)$ defined by $s(x) = x^q$ is continuous.

Remark 4.10. We will accept, without proof, that the functions e^x, $\cos(x)$, $\sin(x)$, $\log(x)$ and other standard functions from calculus are continuous on their domains.

Continuity behaves well when restricting the domain of a function. The proof is, again, follows readily from the definitions and prior facts about limits.

Proposition 4.11. *Suppose $E \subset D \subset \mathbb{R}$, $f : D \to \mathbb{R}$ and $a \in E$. If f is continuous at a, then so is $f|_E$. If f is continuous, the $g = f|_E$ is continuous.*

Conversely, if there is a $\delta > 0$ such that, with $G = D \cap (a - \delta, a + \delta)$, the function $f|_G$ is continuous at a, then f is continuous at a.

That continuity behaves well with respect to the algebraic operations on $\mathbb{R}$ again follows from the corresponding facts about limits.

Proposition 4.12. *Suppose $D \subset \mathbb{R}$, $f, g : D \to \mathbb{R}$ and $a \in D$. If both f and g are continuous at a, then so are*

(i) $f + g$;
(ii) fg; *and*
(iii) $\frac{1}{g}$, *assuming that g is never 0.*

Moreover, if f takes non-negative values and q is a positive rational number, then $h : D \to \mathbb{R}$ defined by $h(x) = f(x)^q$ is continuous at a.

4.1. **Compositions of continuous functions.**

Proposition 4.13. *Suppose $f : D \to E$ and $g : E \to \mathbb{R}$.*

If f is continuous at $a \in D$ and g is continuous at $b = f(a) \in E$, then $g \circ f : D \to \mathbb{R}$ is continuous at a.

If both f and g are continuous, then $g \circ f$ is continuous.

Proof. Let $\epsilon > 0$ be given. There is a $\eta > 0$ such that if $y \in E$ and $|y - b| < \eta$, then $|g(y) - g(b)| < \epsilon$. There is a $\delta > 0$ such that if $x \in D$ and $|a - x| < \delta$, then $|f(x) - f(a)| < \eta$. Hence if $x \in D$ and $|a - x| < \delta$, then $|g(f(x)) - g(f(a))| < \epsilon$.

The second part follows immediately from the first. □

Example 4.14. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2+1}$ is continuous.

Continuity at a point also has a sequential characterization.

Proposition 4.15. *Suppose $f : D \to \mathbb{R}$ and $a \in D$. If f is continuous at a and if (a_n) is a sequence from D which converges to a, then $(f(a_n))$ converges to $f(a)$.*

Proof. First suppose f is continuous at a and (a_n) is a sequence from D which converges to a. In this case, given $\epsilon > 0$ there is a $\delta > 0$ such that if $x \in D$ and $|a - x| < \delta$, then $|f(x) - f(a)| < \epsilon$. There is an $N \in \mathbb{N}^+$ such that if $n \geq N$, then $|a - a_n| < \delta$. Hence, if $n \geq N$, then $|f(a_n) - f(a)| < \epsilon$ and it follows that $(f(a_n))$ converges to $f(a)$.

To prove the converse, suppose f is not continuous at a. In this case there exists an $\eta > 0$ such that for every $\delta > 0$ there is an x such that $x \in D$ and $|x - a| < \delta$, but $|f(x) - f(a)| \geq \eta$. Choosing, for $n \in \mathbb{N}^+$, $\delta = \delta_n = \frac{1}{n}$, there is an $x_n \in D$ such that $|x_n - a| < \frac{1}{n}$, but $|f(x_n) - f(a)| \geq \eta$. By construction (x_n) is a sequence from D which converges to a, but $(f(x_n))$ does not converge to $f(a)$. □

4.2. The extreme and intermediate value theorems.

Lemma 4.16. *If $f : [a,b] \to \mathbb{R}$ is continuous, then f is bounded; i.e., the set $f([a,b]) = \{f(x) : a \leq x \leq b\}$ is a bounded both above and below.*

Proof. We will argue by contradiction to show that f is bounded above. Accordingly, suppose f is not bounded above. In this case, for each $n \in \mathbb{N}$, there is an $x_n \in [a,b]$ such that $f(x_n) \geq n$. There is a subsequence (x_{n_k}) which converges to some y by Theorem 2.43. Since $a \leq x_{n_k} \leq b$, it follows that $y \in [a,b]$. By Proposition 4.15, $(f(x_{n_k}))_k$ converges to $f(y)$, a contradicting the fact that $(f(x_{n_k}))$ is unbounded. □

Theorem 4.17 (Extreme Value Theorem (EVT))**.** *If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists $a \leq y \leq b$ such that $f(y) \geq f(x)$ for all $a \leq x \leq b$.*

Remark 4.18. Under the hypotheses of the theorem, the conclusion says that the range of f, namely the set $S = \{f(x) : a \leq x \leq b\}$ has a largest value. Thus the set S has a maximum and of course this maximum is the lub of S. Informally, the theorem is stated as: a continuous function on a closed bounded interval attains its maximum.

Of course f also attains its minimum.

The maximum and minimum of f are the *extrema* or *extreme values* of f.

Proof. The set S is bounded by Lemma 4.16 and it thus has a least upper bound, say M. For each $n \in \mathbb{N}^+$ there is an $y_n \in S$ such that $M - \frac{1}{n} < y_n \leq M$, by the least property of M. For each n there is an x_n in $[a,b]$ so that $f(x_n) = y_n$. By Proposition 2.43, there is a z and a subsequence (x_{n_k})

of (x_n) which converges to z. Since f is continuous, $(f(x_{n_k}))_k$ converges to $f(z)$. On the other hand, from the construction $(y_n = f(x_n))$ converges to M. Hence $f(z) = M$ and the proof is complete. $\square$

Theorem 4.19 (Intermediate Value Theorem (IVT))**.** *Suppose* $f : [a, b] \to \mathbb{R}$ *is continuous and* $f(a) < f(b)$*. If* $f(a) < k < f(b)$*, then there exists* $a < c < b$ *such that* $f(c) = k$*.*

Proof. Let
$$E = \{x \in [a, b] : f(x) \leq k\}.$$
Note that $a \in E$ and E is bounded above by b. Thus E has a least upper bound $a < c \leq b$. For each $n \in \mathbb{N}^+$ there is a point $x_n \in E$ such that $x_n > c - \frac{1}{n}$. In particular, (x_n) converges to c and $f(x_n) \leq k$. By continuity of f, the sequence $(f(x_n))$ converges to $f(c)$ and moreover, $f(c) \leq k$. In particular, $c < b$. On the other hand, if $t > c$, then $f(t) > k$ as otherwise $t \in E$. Choosing any sequence (t_n) such that $t_n > c$ and (t_n) converges to c, it follows that $(f(t_n))$ converges to $f(c)$ (by continuity of f again) and moreover $f(c) \geq k$. $\square$

Corollary 4.20 (Brower's fixed point theorem)**.** *If* $f : [a, b] \to [a, b]$ *is continuous, then there exists a point* $p \in [a, b]$ *such that* $f(p) = p$*.*

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = x - f(x)$. Observe that $g(a) = a - f(a) \leq 0$ and $g(b) = b - f(b) \geq 0$. Hence, by Theorem 4.19, there is a point p such that $g(p) = 0$. Hence $p - f(p) = 0$ and the corollary is proved. $\square$

Corollary 4.21. *If* $f : [a, b] \to \mathbb{R}$ *is continuous, then there exists* $c \leq d$ *such that* $f([a, b]) = [c, d]$*.*

Proof. Let c and d denote the minimum and maximum values of f which exist by Theorem 4.17. In particular, $f(C) \subset [c, d]$. Moreover, there exists points u and v in $[a, b]$ such that $f(u) = c$ and $f(v) = d$. Assuming $u < v$, given any $c \leq k \leq d$, there is a point $u \leq z \leq v$ such that $f(z) = k$. Hence $f(C) \supset [c, d]$. $\square$

Problem 4.1. Show, if $f : [a, b] \to \mathbb{R}$ is an increasing function and the range of f is an interval, then f is continuous.

4.3. **Open and closed sets - elementary topology of the real line.**

Definition 4.22. The *complement* of a subset C of $\mathbb{R}$ is the set
$$\tilde{C} = \{x \in \mathbb{R} : x \notin C\}.$$
If B is also a subset of C, then
$$B \setminus C = B \cap \tilde{C}.$$
In particular,
$$\tilde{C} = \mathbb{R} \setminus C.$$

Definition 4.23. Given $\delta > 0$, the *δ-neighborhood* of a point $a \in \mathbb{R}$ is the set

$$N_\delta(x) = (a - \delta, a + \delta).$$

A subset O of $\mathbb{R}$ is *open* if for each $p \in O$ there exists a $\delta > 0$ such that $N_\delta(x) \subset O$.

A subset C of $\mathbb{R}$ is *closed* if $\tilde{C}$ is open.

Example 4.24. Given $r > 0$ and $a \in \mathbb{R}$ the set $N_r(a)$ is open.

To prove that $N_r(a)$ is open, let $b \in N_r(a)$ be given. Choose $\delta = r - |a - b| > 0$. Now, if $x \in N_\delta(b)$, then $|x - a| \leq |x - b| + |b - a| < \delta + |a - b| = r$. Thus, $N_\delta(b) \subset N_r(a)$.

Example 4.25. The set $[0, 1) \subset \mathbb{R}$ is not open.

Observe that $0 \in [0, 1)$, but for each $\delta > 0$ we have $N_\delta(0) \not\subset [0, 1)$.

Proposition 4.26. *Suppose I is a set. If for each $i \in I$ the set $O_i \subset \mathbb{R}$ is open, then*

$$O = \cup_{i \in I} O_i$$

is open; i.e., arbitrary unions of open sets are open.

Proof. Let $x \in O$ be given. There is a $j \in I$ such that $x \in O_j$. Since O_j is open and $x \in O_j$, there is a $\delta > 0$ such that $N_\delta(x) \subset O_j$. Since $O_j \subset O$, it follows that $N_\delta(x) \subset O$ and hence O is open. □

Example 4.27. The set $V = \cup_{i=0}^{\infty} (i, i + 1)$ is open.

For any a the sets $(-\infty, a)$ and (a, ∞) are open.

Example 4.28. To see that the set $C = [0, 1]$ is closed, simply observe that $\tilde{C} = (-\infty, 0) \cup (1, \infty)$ is the union of open sets and is hence open.

Proposition 4.29. Suppose $n \in \mathbb{N}^+$. If $O_1, \dots, O_n$ are open sets, then so is

$$O = \cap_{j=1}^{n} O_j.$$

Example 4.30. Let $U_n = (0, 1 + \frac{1}{n})$. Then each U_n is open, but

$$(0, 1] = \cap_{n=1}^{\infty} U_n$$

is not.

Problem 4.2. Prove finite sets are closed.

4.4. **Closed sets.**

Proposition 4.31. *A subset C of $\mathbb{R}$ is closed if and only if C contains all its accumulation points.*

Proof. First suppose a is an accumulation point of C, but $a \notin C$. Because a is an accumulation point, given $\delta > 0$ the set

$$N_\delta(a) \cap C \neq \emptyset.$$

Hence, for every $\delta > 0$, $N_\delta(a) \not\subset \tilde{C}$. Thus $\tilde{C}$ is not open and thus C is not closed.

Conversely, suppose C is not closed in which case $\tilde{C}$ is not open. Hence, there exists a point $a \in \tilde{C}$ such that for every $\delta > 0$,

$$N_\delta(a) \not\subset \tilde{C}.$$

Thus, for every $\delta > 0$,

$$N_\delta(a) \cap C \neq \emptyset.$$

Since $a \notin C$, it follows that a is an accumulation point of C. □

Proposition 4.32. *If C is nonempty, bounded above, and closed, then C contains its least upper bound.*

Proof. The hypotheses imply that C has a least upper bound, say b. Given $\delta > 0$, there is a $c \in C$ such that $b - \delta < c \leq b$. Thus, b is either in C or b an accumulation point of C. By Proposition 4.31, $b \in C$. □

Proposition 4.33. *If C is closed and (a_n) is a sequence from C which converges, then $\lim a_n \in C$.*

Proof. Let $A = \lim a_n$. If $A = a_n$ for some n, then $A \in C$. Otherwise, by Problem 2.10, A is a limit point of C. Since C is closed, Propostion 4.31 implies that $A \in C$. □

Problem 4.3. Prove the converse of Proposition 4.33.

Proposition 4.34. Suppose C is closed and bounded. If (a_n) is a sequence from C, then (a_n) has a subsequence which converges to some point of C.

Proof. Since C is bounded, so is (a_n). Hence, by Theorem 2.43, there is a subsequence (a_{n_k}) of (a_n) which converges to some a. By Proposition 4.33, $a \in C$. □

4.5. **Continuous functions on closed bounded sets.**

Proposition 4.35. *Suppose C is nonempty, closed and bounded. If $f : C \to \mathbb{R}$ is continuous, then $f(C)$ is closed and bounded too.*

Proof. Suppose $f(C)$ is not bounded above. In this case, for each n there exists $y_n \in C$ such that $y_n \geq n$. For each n there is an $x_n \in C$ such that $f(x_n) = y_n$. There is a subsequence (x_{n_k}) of (x_n) converging to some $z \in C$ by Proposition 4.34. By continuity of f, the sequence $(f(x_{n_k}) = y_{n_k})$ is convergent and thus bounded, a contradiction which shows $f(C)$ is in fact bounded above. A similar argument shows $f(C)$ is bounded below.

By Proposition 4.31, to see that $f(C)$ is closed it suffices to show that it contains all its accumulation points. Accordingly, suppose p is an accumulation point of $f(C)$. In particular, there is a sequence (y_n) from $f(C)$ which converges to p. For each n there is an $x_n \in C$ such that $y_n = f(x_n)$. By Proposition 4.34, there is a subsequence (x_{n_k}) which converges to some $z \in C$. By continuity of f, the sequence $(y_{n_k} = f(x_{n_k}))$ converges to $f(z)$. But (y_{n_k}) converges to p. Hence $f(z) = p$ and thus $p \in f(C)$. □

Corollary 4.36 (EVT II)**.** *If C is nonempty, closed and bounded and if $f: C \to \mathbb{R}$ is continuous, then there exists a point $z \in C$ such that $f(z) \geq f(x)$ for every $x \in C$; i.e., f attains its extrema on C.*

The proof is left as an exercise for the gentle reader.

4.6. **Inverse functions.**

Definition 4.37. Given a set A, the function $id_A : A \to A$ defined by $id_A(a) = a$ is called the *identity function.*

Proposition 4.38. *Given a function $f : A \to B$, there exists a function $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$ if and only if f is one-one and onto. Moreover, in this case, g is unique.*

Proof. First suppose that f is one-one and onto. Define $g : B \to A$ as follows. Given $b \in B$ there is a unique $a \in A$ such that $b = f(a)$ (because f is both one-one and onto). Let $g(b) = a$. Then $g(f(a)) = g(b) = a$ and $f(g(b)) = f(a) = b$.

Conversely, suppose there is a g such that both $f \circ g = id_B$ and $g \circ f = id_A$. To prove that f is one-one, suppose $f(x) = f(y)$. Then $x = g(f(x)) = g(f(y)) = y$. To prove that f is onto, let $b \in B$ be given and observe that $f(g(b)) = b$.

Finally, to see that g is unique suppose also that $f \circ h = id_B$. It follows that $g \circ f \circ h = h$ and also $g \circ f \circ h = g \circ id_B = g$. □

Definition 4.39. The function g in Proposition 4.38 (assuming it exists) is called the *inverse of f* and is denoted f^{-1}.

Remark 4.40. Of course if f has an inverse g, then by Proposition 4.38, g has an inverse and $g^{-1} = f$.

Example 4.41. Define $f : [0, \infty) \to [0, \infty)$ by $f(x) = x^2$. From Proposition 1.12 it follows that f is one-one. On the other hand, for each $b > 0$ the Intermediate Value Theorem (Theorem 4.19) implies that $f([0, b]) = [0, b^2]$. Consequently, f is onto and thus has an inverse. Of course, the inverse of f is the square root function.

The functions log and exp are inverses of each other.

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin(x)$ does not have an inverse. However, the function $F : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by $F(x) = \sin(x)$ is one-one and, using the Intermediate Value Theorem (and continuity of sin) it is onto. Thus f has an inverse we call the arcsin. Thus $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ and for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\arcsin(F(x)) = x$ and $F(\arcsin(y)) = y$ for $-1 \leq y \leq 1$.

Theorem 4.42. *If C is nonempty, closed and bounded and if $f : C \to f(C)$ is one-one, then $f^{-1} : f(C) \to C$ is continuous.*

Proof. For notational ease, let $g = f^{-1}$. Fix $w \in f(C)$ and, arguing by contradiction, suppose g is not continuous at w. In this case, there exists an $\eta > 0$ such that for each $n \in \mathbb{N}^+$ there exists $y_n \in f(C)$ such that $|y_n - w| < \frac{1}{n}$, but

$$|g(y_n) - g(w)| \geq \eta. \tag{2}$$

Let $x_n = g(y_n)$. Since (x_n) is a sequence from the closed and bounded set C, it has a subsequence (x_{n_k}) which converges to some $z \in C$ by Proposition 4.34. By continuity of f, the sequence $(f(x_{n_k}) = y_{n_k})$ converges to $f(z)$. But (y_{n_k}) converges to w by construction. Thus $w = f(z)$ so that $z = g(w)$. Hence $(g(y_{n_k}) = x_{n_k})$ converges to $z = g(w)$, contradicting (2). □

4.7. **Uniform continuity.**

Definition 4.43. Given $D \subset \mathbb{R}$, a function $f : D \to \mathbb{R}$ is *uniformly continuous* if for every $\epsilon > 0$ there is a $\delta > 0$ such that if $x, y \in D$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Example 4.44. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is uniformly continuous.

Example 4.45. The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not uniformly continuous.

To show g is not uniformly continuous, choose $\epsilon_0 = 1$. Given $\delta > 0$ choose $x = \frac{1}{\delta}$ and $y = x + \frac{\delta}{2}$. Then $|x - y| < \delta$, but

$$|f(y) - f(x)| = 1 \geq \epsilon_0.$$

Example 4.46. Define $h : [1, \infty) \to \mathbb{R}$ by $h(x) = \sqrt{x}$. To see that h is uniformly continuous, let $\epsilon > 0$ be given. Choose $\delta = 2\epsilon$. If $1 \leq x, y$ and $|x - y| < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &= |\sqrt{x} - \sqrt{y}| \\ &= \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \\ &\leq \frac{|x - y|}{2} \\ &< \frac{\delta}{2} = \epsilon. \end{aligned}$$

Problem 4.4. Define $f : (0, 1) \to \mathbb{R}$ by $f(x) = \frac{1}{x}$. Show f is not uniformly continuous.

Theorem 4.47. *If C is nonempty, closed and bounded and if $f : C \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

Proof. Arguing by contradiction, suppose f is not uniformly continuous. In this case there is an $\epsilon_0 > 0$ such that for every $n \in \mathbb{N}^+$ there exists points $x_n, y_n \in C$ such that

$$|x_n - y_n| < \frac{1}{n}, \tag{3}$$

but

(4) $$|f(x_n) - f(y_n)| \geq \epsilon_0.$$

By Proposition 4.34 there is a point $z \in C$ and a subsequence (x_{n_k}) of (x_n) which converges to z. As an exercise for the reader, use (3) to show that (y_{n_k}) converges to z also.

By continuity of f, the sequences $(f(x_{n_k}))$ and $(f(y_{n_k}))$ both converge to $f(z)$ which contradicts (4). □

Example 4.48. Given $a < b$, the function $f : [a, b] \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.

Similarly, for any $b > 0$, the function $g : [0, b] \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$ is continuous.

Problem 4.5. Suppose $f : [0, \infty) \to \mathbb{R}$ is continuous and there is a $b > 0$ such that $f|_{[b,\infty)}$ is uniformly continuous, then f is uniformly continuous.

Show that $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous. (Compare with Example 4.46.)

Problem 4.6. Show, if $f : [a, b) \to \mathbb{R}$ is continuous and $\lim_{x\to b} f(x)$ exists, then f is uniformly continuous and bounded.

[Suggestion: Let L denote the limit and define $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x)$ if $a \leq x < b$ and $g(b) = L$ and prove g is continuous.]

Proposition 4.49. *If $f : D \to \mathbb{R}$ is uniformly continuous and if (x_n) is a Cauchy sequence from D, then $(f(x_n))$ is a Cauchy sequence.*

The proof is left as an exercise for the gentle reader.

Problem 4.7. Show f in Problem 4.4 is not uniformly continuous.

Problem 4.8. Show if $f : [a, b) \to \mathbb{R}$ is uniformly continuous, then $\lim_{x\to b} f(x)$ exists. Conclude that f is bounded.

4.8. **Lipschitz continuity.**

Definition 4.50. A function $f : D \to \mathbb{R}$ is *Lipschitz continuous* if there exists a C such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in D$.

If C can be chosen such that $0 \leq C < 1$, then f is a *contraction* or *contraction mapping*.

Remark 4.51. If f is Lipschitz continuous, then f is uniformly continuous.

Example 4.52. The function $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is uniformly continuous, but not Lipschitz continuous.

Because f is a continuous function on a closed and bounded set it is uniformly continuous. To see that f is not Lipschitz continuous, let $C > 0$ be given. With $0 < x < C^{-2}$, we have $C < \frac{1}{\sqrt{x}}$ and hence

$$C|0 - x| = Cx < \sqrt{x} = |f(0) - f(x)|.$$

5. Differentiation

5.1. Definitions and examples.

Definition 5.1. Suppose $f : D \to \mathbb{R}$ and $a \in D$ is an accumulation point of D. Define $g : D \setminus \{a\} \to \mathbb{R}$ by

$$g(x) = \frac{f(x) - f(a)}{x - a}.$$

If $\lim_{x\to a} g(x)$ exists, then f is *differentiable at* a and the limit is the *derivative of* f *at* a, denoted $f'(a)$.

Example 5.2. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

To see that f is differentiable at 0, consider g, defined for all real numbers except 0 by

$$g(x) = \frac{f(x) - f(0)}{x - 0} = x \sin(\frac{1}{x}).$$

A routine argument shows $\lim_{x\to 0} g(x) = 0$. Thus f is differentiable at 0 and $f'(0) = 0$.

Example 5.3. Show $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is not differentiable at 0.

Proposition 5.4. *Suppose $f : D \to \mathbb{R}$ and a is an accumulation point of D. If f is differentiable at a, then f is continuous at a.*

Remark 5.5. Example 5.3 shows the converse of the proposition is false.

Proof. Note that

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

exists by hypothesis. Thus,

$$\begin{aligned} 0 =& f'(a) \lim_{x\to a}(x - a) \\ =& \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x - a) \\ =& \lim_{x\to a}[f(x) - f(a)]. \end{aligned}$$

Hence,

$$f(a) = \lim_{x\to a}[f(x) - f(a)] + \lim_{x\to a} f(a) = \lim_{x\to a} f(x),$$

so that the limit on the right hand side exists and is equal $f(a)$. Hence f is continuous at a. □

Definition 5.6. Suppose $f : D \to \mathbb{R}$ and every point of D is an accumulation point of D. If f is differentiable at each $a \in D$, then f is *differentiable*.

Remark 5.7. Often, when discussing differentiation, D is an open interval. If f is differentiable, then we obtain a function $f' : D \to \mathbb{R}$.

Example 5.8. Show, if c is a constant, then $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = c$ is differentiable and $f'(x) = 0$.

Show $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ is differentiable and $f'(x) = 1$.

Show $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is differentiable.

5.2. **Properties of the derivative.**

Theorem 5.9. *Suppose $f, g : D \to \mathbb{R}$ and $a \in D$ is an accumulation point of D. If both f and g are differentiable at a, then*

(i) $f + g$ is differentiable at a and $(f + g)'(a) = f'(a) + g'(a)$;
(ii) fg is differentiable at a and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$; and
(iii) if g is never 0 and $g'(a) \neq 0$, then $h\frac{1}{g}$ is differentiable at a and $h'(a) = -\frac{g'(a)}{g^2(a)}$.

Proof. Here is the proof of item (2). Using properties of limits,

$$\begin{aligned}
f'(a)g(a) + f(a)g'(a) =& g(a) \lim_{x\to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x\to a} g(x) \lim_{x\to a} \frac{f(x) - f(a)}{x - a} + f(a) \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x\to a} \frac{g(x)(f(x) - f(a))}{x - a} + f(a) \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\
&= \lim_{x\to a} \frac{(f(x) - f(a))g(x) + (g(x) - g(a))f(a)}{x - a} \\
&= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}.
\end{aligned}$$

The proofs of the other items are similar and omitted. ☐

Remark 5.10. It follows from the Theorem and Example 5.8 that a rational function is differentiable (on its domain).

Theorem 5.11 (Chain Rule)**.** *Suppose $f : D \to E$, $g : E \to \mathbb{R}$ and $a \in D$ and $b = f(a) \in E$ are accumulation points of D and E respectively. If f is differentiable at a and g is differentiable at b, then $h = g \circ f$ is differentiable at a and $h'(a) = g'(f(a))f'(a)$.*

Proof. The assumption that g is differentiable at a is equivalent to continuity of

$$F(y) = \begin{cases} \frac{g(y) - g(b)}{y - b} & y \neq b \\ g'(b) & y = b, \end{cases}$$

at b. Thus, Proposition 4.13 gives

$$\lim_{x\to a} F(f(x)) = F(b) = g'(b).$$

Note that

$$F(f(x))\frac{f(x)-f(a)}{x-a} = \frac{h(x)-h(a)}{x-a}$$

(for $x \neq a$ of course). Thus, routine properties of limits gives,

$$\begin{aligned}\lim_{x\to a}\frac{g(f(x))-g(b)}{x-a} &= \lim_{x\to a} F(f(x))\frac{f(x)-f(a)}{x-a}\\ &= \lim_{x\to a} F(f(x)) \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\\ &= g'(b)f'(a).\end{aligned}$$

□

Proposition 5.12 (Inverse Function Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous, strictly increasing, and differentiable at $a < c < b$. If $f'(c) \neq 0$, then $f^{-1} : [f(a), f(b)] \to [a,b]$ is differentiable at $f(c)$ and*

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

Proof. For notational ease, let $g = f^{-1}$ and $d = f(c)$. The function

$$F(x) = \begin{cases} \frac{x-c}{f(x)-f(c)} & x \neq c \\ \frac{1}{f'(c)} & x = c \end{cases}$$

is defined and continuous, including at c. Since also $g(y)$ is continuous and the composition of continuous functions is continuous, it follows that

$$\lim_{y\to d} F(g(y)) = F(g(d)) = F(c).$$

Noting that $F(g(y)) = \frac{g(y)-d}{y-d}$ completes the proof. □

Problem 5.1. Define $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ by $f(x) = \sin(x)$. Assuming we know that f is differentiable (and its derivative is $\cos(x)$), the hypotheses of the Inverse Function Theorem are satisfied for f and any point $c \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $g = f^{-1}$ and find $g'(f(c))$.

5.3. **The Mean Value Theorem.**

Definition 5.13. A function $f : D \to \mathbb{R}$ has a *local (relative) minimum* at $c \in D$ if there is a $\delta > 0$ such that if $y \in D$ and $|c-y| < \delta$, then $f(c) \leq f(y)$.

Lemma 5.14. *Suppose $c \in D \subset \mathbb{R}$ and there is an $\eta > 0$ such that $(c-\eta, c+\eta) \subset D$. If $f : D \to \mathbb{R}$ has a local minimum at c and if f is differentiable at c, then $f'(c) = 0$.*

Lemma 5.15 (Rolle's Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous. If $f(a) = f(b)$ and if f is differentiable on the open interval (a,b), then there is a point $a < c < b$ such that $f'(c) = 0$.*

Proof. Without loss of generality, it can be assumed that f is not constant. Since f is continuous on the closed bounded interval $[a,b]$, it attains its extrema. Since $f(a) = f(b)$ and f is not constant, f attains either its maximum or minimum at some point $a < c < b$. From Lemma 5.14 it follows that $f'(c) = 0$. □

Theorem 5.16 (Cauchy Mean Value Theorem). *If $f, g : [a,b] \to \mathbb{R}$ are continuous and differentiable at each point in (a,b), then there is a c with $a < c < b$ so that $(f(b) - f(a))g'(c) = f'(c)(g(b) - g(a))$.*

Proof. Let $F(x) = (f(x) - f(a))(g(b) - g(a)) - (f(b) - f(a))(g(x) - g(a))$. Then $F(a) = F(b) = 0$ and F satisfies the hypotheses of Rolle's Theorem. Hence there is a $a < c < b$ such that $F'(c) = 0$; i.e., $f'(c)(g(b) - g(a)) = f'(x)(g(b) - g(a))$. □

Choosing $g(x) = x$ in the Cauchy Mean Value Theorem captures the usual Mean Value Theorem.

Corollary 5.17 (Mean Value Theorem). *If $f : [a,b] \to \mathbb{R}$ is continuous and differentiable at each point in (a,b), then there is a c with $a < c < b$ so that $f(b) - f(a) = f'(c)(b - a)$.*

Corollary 5.18. *Suppose $f : (u,v) \to \mathbb{R}$ is differentiable.*

The function f is increasing if and only if $f' \geq 0$ (meaning $f'(x) \geq 0$ for all $x \in (a,b)$).

The function f is constant if and only if $f' = 0$.

Example 5.19. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x + 2x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

Then $f'(0) = \frac{1}{2} > 0$, but there is no interval properly containing 0 on which f is increasing. Indeed, for $n \in \mathbb{N}^+$, let

$$x_n = \frac{1}{2n\pi}$$

and note that $f'(x_n) = -1 < 0$ which implies there is no interval properly containing 0 on which $f' \geq 0$.

Problem 5.2. Suppose $f : [a,b] \to \mathbb{R}$ is differentiable. Show, if $f' > 0$ (so $f'(x) > 0$ for all $x \in [a,b]$), then f is strictly increasing. (In particular, the hypotheses of the Inverse Function Theorem are satisfied).

Show the result remains true if $f'(x) > 0$ for all except possibly one $x \in [a,b]$.

5.4. **Further topics.**

Theorem 5.20 (Taylor's Theorem). *Let $I = (u,v) \subset \mathbb{R}$ be an open interval, $n \in \mathbb{N}$, and suppose $f : I \to \mathbb{R}$ is $(n+1)$ times differentiable. If $u < a < b <$*

v, then there is a c such that $a < c < b$ and

$$f(b) = \sum_{j=0}^{n} \frac{f^{(j)}(a)(b-a)^j}{j!} + \frac{f^{(n+1)}(c)(b-a)^{n+1}}{(n+1)!}.$$

Proof. Define $R_n : I \to \mathbb{R}$ by

$$R_n(x) = f(b) - \sum_{j=0}^{n} \frac{f^{(j)}(x)(b-x)^j}{j!}.$$

There is a K so that $R_n(a) = K\frac{(b-a)^{n+1}}{(n+1)!}$ and the goal is to prove there is a $a < c < b$ such that $K = f^{(n+1)}(c)$.

Let

$$\varphi(x) = R_n(x) - K\frac{(b-x)^{n+1}}{(n+1)!}.$$

Note that $\varphi : [a,b] \to \mathbb{R}$ is continuous and differentiable on (a,b). Moreover, $\varphi(a) = 0 = \varphi(b)$. Thus, by the MVT, there is a $a < c < b$ such that $\varphi'(c) = 0$. Since,

$$\varphi'(x) = -f^{(n+1)}(x)\frac{(b-x)^n}{n!} + K\frac{(b-x)^n}{n!},$$

it follows that

$$0 = (-f^{(n+1)}(c) + K)\frac{(b-c)^n}{n!}.$$

The conclusion of the theorem follows. □

Proposition 5.21 (A version of L'hopitals rule)**.** *Let $I = (a,b)$ and $f, g : I \to \mathbb{R}$ and suppose*

(i) *both f and g are differentiable;*

(ii)

$$\lim_{x\to a+} f(x) = 0 = \lim_{x\to a^+} g(x); \text{ and;}$$

(iii) *both g and g' are never 0.*

If

$$\lim_{x\to a+} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. The functions f and g extend to be continuous on $[a,b)$ by defining $f(a) = g(a) = 0$.

Let

$$L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Given $\epsilon > 0$ there is a $\delta > 0$ such that if $a < y < a + \delta$, then

$$|L - \frac{f'(y)}{g'(y)}| < \epsilon.$$

From the Cauchy mean value theorem and hypothesis (iii), given $a < x < a + \delta$ there is a $a < c < x$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f(x)}{g(x)}.$$

Thus, if $a < x < a + \delta$, then,

$$|L - \frac{f(x)}{g(x)}| = |L - \frac{f'(c)}{g'(c)}| < \epsilon.$$

□

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